

AN INVESTMENT MODEL WITH SWITCHING COSTS AND THE OPTION TO ABANDON

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Abstract

We consider an investment project that operates within a random environment and yields a payoff rate that is a function of a stochastic economic indicator such as the price of or the demand for the project's output commodity. We assume that the investment project can operate in two modes, an "open" one and a "closed" one. The transitions from one operating mode to the other one are costly and immediate, and form a sequence of decisions made by the project's management. We also assume that the project can be permanently abandoned at a discretionary time and at a constant sunk cost. The objective of the project's management is to maximise the expected discounted payoff resulting from the project's management over all switching and abandonment strategies. We derive the explicit solution to this stochastic control problem that involves impulse control as well as discretionary stopping. It turns out that this has a rather rich structure and the optimal strategy can take eight qualitatively different forms, depending on the problems data.

Keywords and phrases. Stochastic impulse control, optimal switching, discretionary stopping, real options.

AMS (2010) Subject Classifications. Primary 93E20; secondary 60G40, 90B50.

1 Introduction

Optimal sequential switching is an area of stochastic control that emerged from financial economics in the context of real options (see Dixit and Pindyck [5] and Trigeorgis [27]).

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Its numerous applications include the optimal scheduling of production in a real asset such as a power plant that can operate in distinct modes, say “open” and “closed”, as well as the optimal timing of sequentially investing and disinvesting, e.g., in a given stock. The references Bayraktar and Egami [1], Brekke and Øksendal [2], Bronstein and Zervos [3], Carmona and Ludkovski [4], Djehiche, Hamadène and Popier [7], Duckworth and Zervos [8], El Asri [9], El Asri and Hamadène [10], Elie and Kharroubi [11], Gassiat, Kharroubi and Pham [12], Guo and Tomecek [13], Hamadène and Jeanblanc [14], Hamadène and Zhang [15], Johnson and Zervos [17], Lumley and Zervos [20], Ly Vath and Pham [21], Pham [22], Pham, Ly Vath and Zhou [23], René, Campi, Langrené and H. Pham [24], Song, Yin and Zhang [25], Tang and Yong [26], Tsekrekos and Yannacopoulos [28], Zervos, Johnson and Alazemi [30], Zhang and Zhang [31], and Zhang [32] provide an alphabetically ordered list of important contributions in the area.

In this paper, we derive the complete solution to a problem of optimal sequential switching that incorporates an additional permanent abandonment option. To fix ideas, we consider an investment project that operates within a random environment and yields a payoff rate that is a function of a stochastic economic indicator such as the price of or the demand for the project’s output commodity. We model this economic indicator by the geometric Brownian motion given by

$$dX_t = bX_t dt + \sqrt{2}\sigma X_t dW_t, \quad X_0 = x > 0, \quad (1)$$

where b and $\sigma \neq 0$ are given constants and W is a standard Brownian motion. We assume that the investment project can operate in two modes, an “open” one and a “closed” one. The transitions from one operating mode to the other one are immediate and form a sequence of decisions made by the project’s management. We use a process Z with values in $\{0, 1\}$ to model such a sequence of decisions. In particular, we assume that $Z_t = 1$ (resp., $Z_t = 0$) if the project is “open” (resp., “closed”) at time t . We also denote by $z \in \{0, 1\}$ the project’s mode at time 0, so that $Z_0 = z$. The stopping times at which the jumps of Z occur are the intervention times at which the project’s operating mode is changed. We assume that the project can be permanently abandoned at a stopping time τ , which is an additional decision variable. With each admissible strategy (Z, τ) , we associate the performance criterion

$$J_{z,x}(Z, \tau) = \mathbb{E} \left[\int_0^\tau e^{-rs} h(X_s) Z_s ds - \sum_{j=1}^{\infty} e^{-rT_j^1} K_1 \mathbf{1}_{\{T_j^1 \leq \tau\}} - \sum_{j=1}^{\infty} e^{-rT_j^0} K_0 \mathbf{1}_{\{T_j^0 \leq \tau\}} - e^{-r\tau} K \right], \quad (2)$$

where (T_j^1) (resp., (T_j^0)) is the sequence of times at which Z jumps from 0 to 1 (resp., from 1 to 0). Here, $h :]0, \infty[\rightarrow \mathbb{R}$ models the running payoff resulting from the investment project while this is in its “open” operating mode.¹ The constants $K_1 > 0$ and $K_0 > 0$ are the costs

¹Using a trivial re-parametrisation, we can allow for the project to yield a constant payoff rate while it is in its “closed” mode (see Remark 1).

resulting from “switching” the project from its “closed” mode to its “open” one and vice versa, whereas $K \in \mathbb{R}$ is the cost resulting from the decision to permanently abandon it. Note that we allow for K to be negative, which corresponds to a situation where capital can be recovered at abandonment.² Also, on the event $\{T_j^\ell = \tau\}$, $\ell = 1, 0$, a cost of $K_\ell + K$ is incurred at time T_j^ℓ , which corresponds to the possibility that the project’s operating mode can be switched just before the project is permanently abandoned.³ The objective is to maximise the performance criterion $J_{z,x}$ over the set Π_z of all admissible strategies (Z, τ) . Accordingly, we define the value function v by

$$v(z, x) = \sup_{(Z, \tau) \in \Pi_z} J_{z,x}(Z, \tau), \quad \text{for } (z, x) \in \{0, 1\} \times]0, \infty[. \quad (3)$$

The model that we study is the natural extension of Duckworth and Zervos [8] that arises by incorporating a permanent abandonment option. A special case of this model and its applications in real options are extensively discussed in Dixit and Pindyck [5, Section 7.2] using heuristic arguments and numerical examples. The related special case that arises if $X = W$, $h(x) = x$ and $K > 0$ was solved by Zervos [29]. Furthermore, the existence of an optimal strategy in a more general context with finite time horizon was established by Djehiche and Hamadène [6] using a system of Snell envelopes as well as viscosity solutions. Despite its importance for applications and the fact that there is an extensive literature on problems combining stochastic control with optimal stopping, there are no further references addressing this problem, to the best of our knowledge. Perhaps, this is due to the substantial complexity of the problem.

We derive the complete solution to the problem that we study in an explicit form by solving its Hamilton-Jacobi-Bellman (HJB) equation that takes the form of a pair of coupled quasi-variational inequalities. In particular, we identify the five regions that partition the state space $\{0, 1\} \times]0, \infty[$ and characterise the optimal strategy, namely, the “production” region, the “waiting” region, the “switch in” region, the “switch out” region and the “abandonment” region. It turns out that the qualitative nature of the problem’s solution is surprising rich and can take eight different forms, depending on the problem data.

The value that may be added by waiting before implementing a certain investment decision is a central feature of the real option theory. In some of the cases that arise in our analysis below, *value may be added by waiting before choosing one of two investment actions of a qualitatively different nature, one partially reversible and one totally irreversible*. With the exception of the related special case studied by Zervos [29], we are not aware of other models that have been studied in the real option literature exhibiting this feature. For instance, in Case II.3 in Section 4.2 (see also Figure 6), the part of the “production” region

²For the same reason, it would make sense in some economic applications to allow for at least K_0 to be negative, as long as $K_1 + K_0 > 0$. However, such a relaxation would add most significant complexity and would result in a substantially longer paper.

³Although this setting is convenient for the problem’s formulation, switching followed by immediate abandonment is never optimal due to the strict positivity of K_ℓ , $\ell = 1, 0$.

identified by the set $\{1\} \times]\delta, \gamma[$ separates the “abandonment” region from the “switch out” region. In this case, if the initial condition of the state process is in this part of the state space, then it is optimal to take no action before committing to *either* enter a perpetual cycle of operating the investment project by optimally switching it between its two modes *or* permanently abandoning the project, depending on whether the economic indicator X first rises to the level γ or first drops to the level δ . Furthermore, the investment project has infinite lifetime if the initial condition of the state process is in $\{1\} \times [\gamma, \infty[\cup \{0\} \times]0, \infty[$ and finite lifetime with strictly positive probability otherwise. The situation becomes more dramatic in Case III.2 in Section 4.3 (see also Figure 8). In this case, the part of the “production” region identified by the set $\{1\} \times]\delta, \gamma[$ separates the “abandonment” region from the “switch out” region, while the whole “waiting” region $\{0\} \times]\zeta, \alpha[$ separates the “abandonment” region from the “switch in” region. If the initial condition of the state process is in this part of the “production” region (resp., in the “waiting” region), then it is optimal to take no action before committing to *either* switch the investment project to its “closed” mode *or* permanently abandon it (resp., *either* switch the investment project to its “open” mode *or* permanently abandon it). Contrary to the previous case, the investment project’s lifetime is always finite with strictly positive probability, and with probability 1 if $\mu - \sigma^2 \leq 0$.

The paper is organised as follows. We formulate the stochastic optimisation problem that we solve in Section 2. In Section 3, we consider the problem’s HJB equation, we discuss how it characterises the five regions that determine the optimal strategy and we recall some related implications of the assumptions we make. We present the explicit solution to the stochastic control problem in Section 4. Here, we organise the eight cases that arise in three groups based on the analytical affinity of the different cases. To simplify the exposition of our main results, we collect most proofs in two appendixes.

2 Problem formulation

We build the model that we study on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions and supporting a standard one-dimensional (\mathcal{F}_t) -Brownian motion W . We denote by \mathcal{Z} the family of all (\mathcal{F}_t) -adapted finite variation càglàd processes Z with values in $\{0, 1\}$, and by \mathcal{S} the set of all (\mathcal{F}_t) -stopping times.

As we have discussed in the introduction, we consider an investment project that operates within a random environment and yields a payoff rate that is a function of a stochastic economic indicator that is modelled by the geometric Brownian motion given by (1). We assume that the investment project can operate in two modes, an “open” one and a “closed” one. We use a process $Z \in \mathcal{Z}$ to model such a sequence of decisions: $Z_t = 1$ (resp., $Z_t = 0$) if the project is “open” (resp., “closed”) at time t . We also denote by $z \in \{0, 1\}$ the project’s mode at time 0, so that $Z_0 = z$. The stopping times at which the jumps of Z occur are the

intervention times at which the project's operating mode is changed. If we define recursively

$$T_1^1 = \inf \{t \geq 0 \mid \Delta Z_t = 1\}, \quad T_1^0 = \inf \{t \geq 0 \mid \Delta Z_t = -1\},$$

$$T_{j+1}^1 = \inf \{t > T_j^1 \mid \Delta Z_t = 1\} \quad \text{and} \quad T_{j+1}^0 = \inf \{t > T_j^0 \mid \Delta Z_t = -1\}, \quad \text{for } j \geq 1,$$

where $\Delta Z_t = Z_{t+} - Z_t$ and we adopt the usual convention that $\inf \emptyset = \infty$, then T_j^1 (resp., T_j^0) are the (\mathcal{F}_t) -stopping times at which the project is switched from “closed” to “open” (resp., from “open” to “closed”). We also assume that the project can be permanently abandoned at an (\mathcal{F}_t) -stopping time τ . We define the set of all admissible strategies to be

$$\Pi_z = \{(Z, \tau) \mid Z \in \mathcal{Z}, Z_0 = z, \text{ and } \tau \in \mathcal{S}\}.$$

With each admissible strategy $(Z, \tau) \in \Pi_z$, we associate the performance criterion given by (2). The objective is to maximise the performance criterion $J_{z,x}$ over Π_z . Accordingly, we define the value function v by (3).

For the resulting optimisation problem to be well-posed in the sense that there are no integrability problems and there are no admissible strategies with payoff equal to ∞ , we make the following assumption.

Assumption 1 The running payoff function $h :]0, \infty[\rightarrow \mathbb{R}$ is right-continuous and increasing, $\lim_{x \rightarrow \infty} h(x) = \infty$, and

$$\mathbb{E} \left[\int_0^\infty e^{-rt} |h(X_t)| dt \right] < \infty \quad (4)$$

for every initial condition $x > 0$. Furthermore, $K_1, K_0 > 0$ and $K \in \mathbb{R}$. ■

Remark 1 To simplify the exposition, we have assumed that the investment project yields zero payoff while it is in its “closed” mode. In view of the calculation

$$J_{z,x}(Z, \tau) = \mathbb{E} \left[\int_0^\tau e^{-rs} [\bar{h}(X_s) Z_s - C(1 - Z_s)] ds \right. \\ \left. - K_1 \sum_{j=1}^\infty e^{-rT_j^1} \mathbf{1}_{\{T_j^1 \leq \tau\}} - K_0 \sum_{j=1}^\infty e^{-rT_j^0} \mathbf{1}_{\{T_j^0 \leq \tau\}} - e^{-r\tau} \bar{K} \right] + \frac{C}{r},$$

where C is a constant, $\bar{h} = h - C$ and $\bar{K} = K + \frac{C}{r}$, we can see that allowing for a constant payoff rate while the project is in its “closed” mode can be accommodated trivially in the model that we study. ■

3 The Hamilton-Jacobi-Bellman (HJB) equation

In view of standard stochastic control theory that has been developed and used in references we have discussed in the introduction, we expect that the value function of the problem we study is given by

$$v(1, \cdot) = w_1 \quad \text{and} \quad v(0, \cdot) = w_0, \quad (5)$$

where the functions $w_1, w_0 :]0, \infty[\rightarrow \mathbb{R}$ satisfy the coupled quasi-variational inequalities

$$\max \left\{ \sigma^2 x^2 w_1''(x) + bxw_1'(x) - rw_1(x) + h(x), w_0(x) - w_1(x) - K_0, -w_1(x) - K \right\} = 0, \quad (6)$$

$$\max \left\{ \sigma^2 x^2 w_0''(x) + bxw_0'(x) - rw_0(x), w_1(x) - w_0(x) - K_1, -w_0(x) - K \right\} = 0, \quad (7)$$

as well as appropriate growth conditions (see Zervos [29, Theorem 1] for a general verification theorem). In view of the heuristics explaining the structure of this HJB equation, the state space $\{0, 1\} \times]0, \infty[$ splits into five pairwise disjoint regions⁴:

(i) *The “production” region $\{1\} \times \mathcal{P}$, where \mathcal{P} is an open subset of $]0, \infty[$.* Whenever the project is in its “open” mode and the process X takes values in \mathcal{P} , it is optimal to keep the project in its “open” mode, which is associated with production. In particular, \mathcal{P} is the set in which the function w_1 satisfies the ODE

$$\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) + h(x) = 0. \quad (8)$$

(ii) *The “waiting” region $\{0\} \times \mathcal{W}$, where \mathcal{W} is an open subset of $]0, \infty[$.* If the project is in its “closed” mode and the process X takes values in \mathcal{W} , then it is optimal to take no action, namely, keep the project on standby. The set \mathcal{W} is characterised by the requirement that w_0 satisfies the ODE

$$\sigma^2 x^2 w''(x) + bxw'(x) - rw(x) = 0. \quad (9)$$

(iii) *The “switch out” region $\{1\} \times \mathcal{S}_{\text{out}}$, where \mathcal{S}_{out} is a closed subset of $]0, \infty[$.* If the project is in its “open” mode, then it is optimal to switch it to its “closed” mode as soon as X takes values in \mathcal{S}_{out} . The set \mathcal{S}_{out} is characterised by the identity

$$w_1(x) = w_0(x) - K_0 \quad \text{for all } x \in \mathcal{S}_{\text{out}}. \quad (10)$$

(iv) *The “switch in” region $\{0\} \times \mathcal{S}_{\text{in}}$, where \mathcal{S}_{in} is a closed subset of $]0, \infty[$.* It is optimal to switch the project from its “closed” to its “open” mode as soon as X takes values in \mathcal{S}_{in} . In this case,

$$w_0(x) = w_1(x) - K_1 \quad \text{for all } x \in \mathcal{S}_{\text{in}}. \quad (11)$$

⁴In the description of the five possible regions, we characterise subsets of $]0, \infty[$ as open or closed relative to the topology on $]0, \infty[$ that is the trace of the usual topology on \mathbb{R} , for instance, $]0, a] =]0, \infty[\setminus]a, \infty[$ and $[a, \infty[=]0, \infty[\setminus]0, a[$ are closed sets.

(v) The “abandonment” region $\{0\} \times \mathcal{A}_0 \cup \{1\} \times \mathcal{A}_1$, where $\mathcal{A}_0, \mathcal{A}_1$ are closed subsets of $]0, \infty[$. It is optimal to abandon permanently the project as soon as the state process hits the abandonment region. Accordingly,

$$w_i(x) = -K \quad \text{for all } x \in \mathcal{A}_i \text{ and } i = 0, 1. \quad (12)$$

The tactics associated with these regions exhaust all possible control actions. Therefore,

$$\mathcal{P} \cup \mathcal{S}_{\text{out}} \cup \mathcal{A}_1 = \mathcal{W} \cup \mathcal{S}_{\text{in}} \cup \mathcal{A}_0 =]0, \infty[.$$

We will solve the control problem that we study by identifying these regions and deriving appropriate explicit solutions to the HJB equation (6)–(7). To this end, we will use the following facts. It is well-known that the general solution to the Euler’s ODE (9) is given by

$$w(x) = Ax^m + Bx^n, \quad (13)$$

for some constants $A, B \in \mathbb{R}$, where the constants $m < 0 < n$ are defined by

$$m, n = \frac{1}{2\sigma^2} \left[\sigma^2 - b \mp \sqrt{(b - \sigma^2)^2 + 4\sigma^2 r} \right].$$

If $h :]0, \infty[\rightarrow \mathbb{R}$ is a function satisfying the integrability condition in (4), then a particular solution to the ODE (8) is the function $R_h :]0, \infty[\rightarrow \mathbb{R}$ given by

$$\begin{aligned} R_h(x) &= \frac{1}{\sigma^2(n - m)} \left[x^m \int_0^x s^{-m-1} h(s) ds + x^n \int_x^\infty s^{-n-1} h(s) ds \right] \\ &= \mathbb{E} \left[\int_0^\infty e^{-rs} h(X_s) ds \right]. \end{aligned} \quad (14)$$

A straightforward calculation reveals that

$$R'_h(x) = \frac{1}{\sigma^2(n - m)} \left[mx^{m-1} \int_0^x s^{-m-1} h(s) ds + nx^{n-1} \int_x^\infty s^{-n-1} h(s) ds \right]. \quad (15)$$

Furthermore, for a choice of h as in Assumption 1,

$$R_h \text{ is increasing,} \quad (16)$$

$$h(0) := \lim_{x \downarrow 0} h(x) = r \lim_{x \downarrow 0} R_h(x) \quad \text{and} \quad \lim_{x \rightarrow \infty} R_h(x) = \infty, \quad (17)$$

$$\lim_{T \rightarrow \infty} e^{-rT} \mathbb{E} \left[|R_h(X_T)| \right] = 0 \quad (18)$$

$$\text{and} \quad \mathbb{E} \left[\int_0^T e^{-2rt} X_t^2 |R'_h(X_t)|^2 dt \right] < \infty \quad \text{for all } T > 0. \quad (19)$$

All of these claims regarding the function R_h as well as several more general results can be found in Knudsen, Meister and Zervos [18, Section 4], Johnson and Zervos [16], and Lamberton and Zervos [19, Section 4].

4 The solution to the control problem

We now derive the solution to the stochastic control problem formulated in Section 2 by identifying the sets \mathcal{P} , \mathcal{W} , \mathcal{S}_{out} , \mathcal{S}_{in} , \mathcal{A}_1 , \mathcal{A}_0 we have discussed in the previous section and deriving appropriate solutions to the HJB equation (6)–(7) using (8)–(12). To this end, we first note that, if the investment project is in its “open” mode at time 0 and is never switched to its “closed” mode or abandoned, then it will yield a total expected discounted payoff equal to $R_h(x)$ (see (14)). On the other hand, if the project is “closed” at time 0 and is never switched to its “open” operating mode or abandoned, then it will yield 0 total expected discounted payoff. Since R_h is increasing and $\lim_{x \rightarrow \infty} R_h(x) = \infty$ (see (16) and (17)), it should be optimal to operate the project in its “open” mode whenever the process X takes sufficiently high values. It follows that there exists $M > 0$ such that

$$]M, \infty[\subseteq \mathcal{P} \quad \text{and} \quad]M, \infty[\subseteq \mathcal{S}_{\text{in}}.$$

If $\mathcal{A}_1 \neq \emptyset$ (resp. $\mathcal{A}_0 \neq \emptyset$), then $\mathcal{A}_1 =]0, \delta]$ (resp., $\mathcal{A}_0 =]0, \zeta]$) for some $\delta > 0$ (resp., $\zeta > 0$) because R_h is increasing. Furthermore, in view of the smoothness of a solution to the HJB equation (6)–(7) that is required to identify it with the control problem’s value function and the analysis in the previous section, we expect that the “abandonment” region does not have any common boundary points with either the “switch in” region or the “switch out” region.

In light of these observations, we will show that the production and the waiting regions \mathcal{P} and \mathcal{W} have the general forms

$$\mathcal{P} =]\delta, \gamma[\cup]\beta, \infty[\quad \text{and} \quad \mathcal{W} =]\zeta, \alpha[, \quad (20)$$

for some $0 \leq \delta \leq \gamma \leq \beta < \infty$ and $0 \leq \zeta \leq \alpha < \infty$ (see Figures 1-8), where we adopt the usual convention that, e.g., $]0, 0[= \emptyset$. In view of the solutions to the ODEs (8), (9) given in the previous section, the solution to the HJB equation (6)–(7) should be such that

$$w_1(x) = \begin{cases} R_h(x), & \text{for all } x \in]0, \infty[, \text{ if } \delta = \gamma = \beta = 0 \\ Ax^m + R_h(x), & \text{for all } x \in]\beta, \infty[, \text{ if } \gamma < \beta \text{ or } 0 < \delta = \gamma = \beta \\ \Gamma_1 x^m + \Gamma_2 x^n + R_h(x), & \text{for all } x \in]\delta, \gamma[, \text{ if } 0 < \delta < \gamma < \beta \end{cases} \quad (21)$$

and

$$w_0(x) = \begin{cases} Bx^n, & \text{for all } x \in]0, \alpha[, \text{ if } \zeta = 0 < \alpha \\ \Delta_1 x^m + \Delta_2 x^n, & \text{for all } x \in]\zeta, \alpha[, \text{ if } 0 < \zeta < \alpha \end{cases}, \quad (22)$$

for some constants A , Γ_1 , Γ_2 , B , Δ_1 and Δ_2 because these are the only choices that are consistent with the requirements of the verification theorem that we will use to identify the solution to (6)–(7) with the control problem’s value function.

To determine free-boundary points such as δ , γ , β , ζ , α appearing in (20) and constants such as A , Γ_1 , Γ_2 , B , Δ_1 , Δ_2 appearing in (21)–(22), we will use the C^1 continuity that we

expect the functions w_1, w_0 to have. In particular, we will require that w_1, w_0 should be C^1 at every boundary point separating any two of the five regions. Using the expressions (14), (15) and the identity $\sigma^2 mn = -r$, we will then derive appropriate systems of equations for the unknown parameters. We will only provide the results of these calculations because they are straightforward to replicate.

We have organised the presentation of the possible cases arising by splitting them in three groups. Group I includes the cases in which it is not optimal to switch or abandon the project if this is in its “open” mode. Group II contains all cases where it may be optimal to switch or abandon the project if this is in its “open” mode but abandonment is not optimal if the project is in its “closed” mode. Finally, Group III includes all remaining cases. To make the presentation easier to follow, we develop the proofs in Appendix II.

4.1 Group I: taking action is not optimal whenever the project is in its “open” operating mode ($\mathcal{P} =]0, \infty[$)

All cases in this group are such that $\mathcal{P} =]0, \infty[$ and are associated with a solution to the HJB equation (6)–(7) such that

$$w_1(x) = R_h(x) \quad \text{for all } x > 0. \quad (23)$$

Case I.1 (Figure 1) In this case, it is optimal to immediately switch the investment project to its “open” mode if it is originally “closed”. Accordingly,

$$\mathcal{P} = \mathcal{S}_{\text{in}} =]0, \infty[\quad \text{and} \quad \mathcal{W} = \mathcal{S}_{\text{out}} = \mathcal{A}_0 = \mathcal{A}_1 = \emptyset,$$

and the functions w_1 and w_0 given by (23) and

$$w_0(x) = R_h(x) - K_1, \quad \text{for } x > 0, \quad (24)$$

should satisfy the HJB equation (6)–(7).

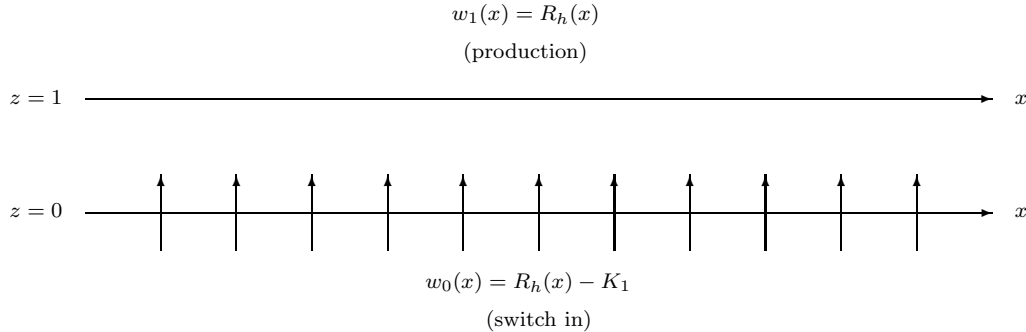


Figure 1. Illustration of the regions determining the optimal strategy in the context of Case I.1

Lemma 1 *The increasing functions w_1, w_0 defined by (23), (24) satisfy the HJB equation (6)–(7) if and only if*

$$\max\{rK_1, rK_1 - rK\} \leq h(0).$$

Case I.2 (Figure 2) In this case, it is optimal to switch the investment project to its “open” mode if it is originally “closed” as long as the process X takes sufficiently high values. In particular, there exists a boundary point $\alpha > 0$ such that, if the project starts in its “closed” mode, then it is optimal to wait for all long as X takes values strictly less than α and switch the project to its “open” mode as soon as X takes a value exceeding α . Accordingly,

$$\mathcal{P} =]0, \infty[, \quad \mathcal{W} =]0, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{S}_{\text{out}} = \mathcal{A}_0 = \mathcal{A}_1 = \emptyset.$$

In view of (11) and (21)–(22), the functions w_1 and w_0 given by (23) and

$$w_0(x) = \begin{cases} Bx^n, & \text{if } x < \alpha \\ R_h(x) - K_1, & \text{if } x \geq \alpha \end{cases} \quad (25)$$

should satisfy the HJB equation (6)–(7).

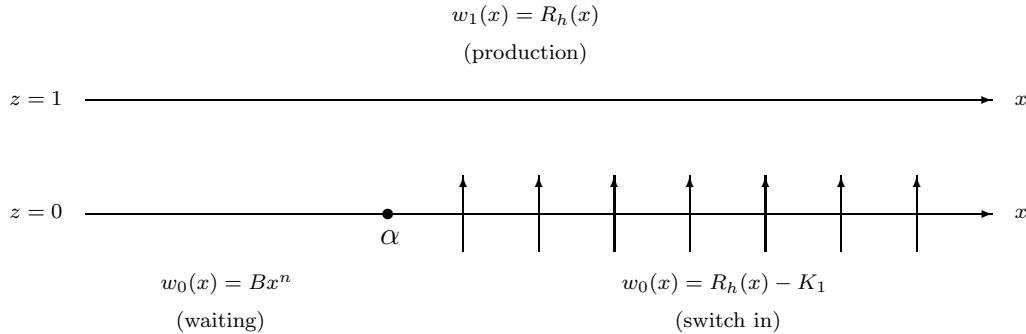


Figure 2. Illustration of the regions determining the optimal strategy in the context of Case I.2

The requirement that w_0 should be C^1 at α yields the expressions

$$B = \frac{1}{\sigma^2(n-m)} \int_{\alpha}^{\infty} s^{-n-1} [h(s) - rK_1] ds \quad (26)$$

$$\text{and} \quad \int_0^{\alpha} s^{-m-1} [h(s) - rK_1] ds = 0. \quad (27)$$

Lemma 2 *Equation (27) has a unique solution $\alpha > 0$ and the functions w_1, w_0 defined by (23), (25), for $B > 0$ given by (26), are increasing and satisfy the HJB equation (6)–(7) if and only if*

$$0 \leq K \quad \text{and} \quad \max\{-rK_0, -rK\} \leq h(0) < rK_1.$$

Case I.3 (Figure 3) This case differs from the previous one by the fact that abandoning the investment project if it is in its “closed” mode and the process X takes values below a given threshold level ζ becomes optimal. Accordingly,

$$\mathcal{P} =]0, \infty[, \quad \mathcal{A}_0 =]0, \zeta], \quad \mathcal{W} =]\zeta, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{S}_{\text{out}} = \mathcal{A}_1 = \emptyset,$$

and, in view of (11)–(12) and (21)–(22), the required solution to the HJB equation (6)–(7) should be given by the function w_1 defined by (23) and the function w_0 defined by

$$w_0(x) = \begin{cases} -K, & \text{if } x \leq \zeta \\ \Delta_1 x^m + \Delta_2 x^n, & \text{if } \zeta < x < \alpha \\ R_h(x) - K_1, & \text{if } x \geq \alpha \end{cases}. \quad (28)$$

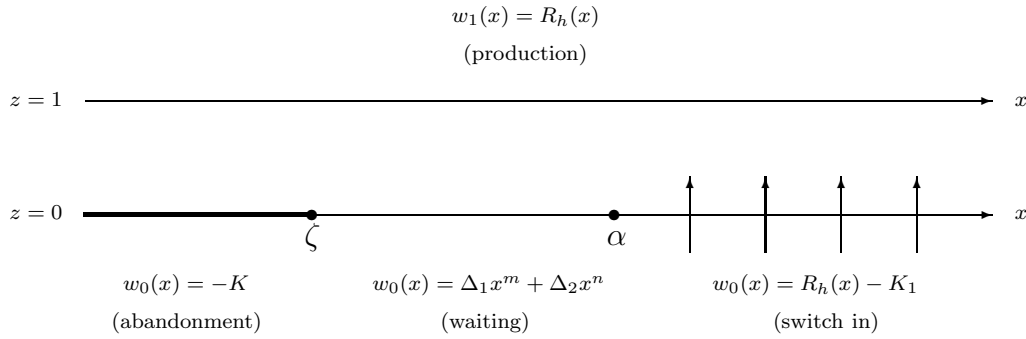


Figure 3. Illustration of the regions determining the optimal strategy in the context of Case I.3

To determine the free-boundary points ζ , α and the parameters Δ_1 , Δ_2 , we require that w_0 should be C^1 , which yields the expressions

$$f_1(\zeta, \alpha) := m \int_0^\alpha s^{-m-1} [h(s) - rK_1] ds - rK\zeta^{-m} = 0, \quad (29)$$

$$f_2(\zeta, \alpha) := n \int_\alpha^\infty s^{-n-1} [h(s) - rK_1] ds + rK\zeta^{-n} = 0, \quad (30)$$

$$\Delta_1 = \frac{rK\zeta^{-m}}{\sigma^2 m(n-m)} \quad \text{and} \quad \Delta_2 = -\frac{rK\zeta^{-n}}{\sigma^2 n(n-m)}. \quad (31)$$

Lemma 3 *The system of equations (29)–(30) has a unique solution (ζ, α) such that $0 < \zeta < \alpha$ and the functions w_1 , w_0 defined by (23), (28), for $\Delta_1 > 0$, $\Delta_2 > 0$ given by (31), are increasing and satisfy the HJB equation (6)–(7) if and only if*

$$K < 0 \quad \text{and} \quad -rK \leq h(0) < rK_1 - rK.$$

4.2 Group II: taking action may be optimal if the project is in its “open” mode but abandonment is not optimal whenever the project is in its “closed” operating mode ($\mathcal{P} \neq]0, \infty[$ and $\mathcal{A}_0 = \emptyset$)

We now consider cases that complement the ones in the previous group and are characterised by the non-optimality of abandonment whenever the project is in its “closed” mode. In all of these cases, $\mathcal{W} =]0, \alpha[$ and $\mathcal{S}_{\text{in}} = [\alpha, \infty[$. Otherwise, the cases are differentiated by the arrangement of the optimal tactics whenever the project is in its “open” mode.

Case II.1 (Figure 4) In this case, sequential switching of the investment project from “open” to “closed” and vice versa is optimal, and abandonment is not part of the optimal strategy. Whenever the project is in its “open” (resp., “closed”) mode, it is optimal to stay there for as long as the process X takes values above (resp., below) a given threshold β (resp., α) and switch to its “closed” (resp., “open”) mode as soon as X takes values below (resp., above) the threshold β (resp., α). Of course, for such a strategy to be well-defined, we must have $\beta < \alpha$. Accordingly,

$$\mathcal{S}_{\text{out}} =]0, \beta], \quad \mathcal{P} =]\beta, \infty[, \quad \mathcal{W} =]0, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{A}_0 = \mathcal{A}_1 = \emptyset.$$

In view of (10)–(11) and (21)–(22), we can see that the required solution to the HJB equation (6)–(7) should be given by the functions defined by

$$w_1(x) = \begin{cases} Bx^n - K_0, & \text{if } x \leq \beta \\ Ax^m + R_h(x), & \text{if } x > \beta \end{cases} \quad (32)$$

$$\text{and } w_0(x) = \begin{cases} Bx^n, & \text{if } x < \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{cases}. \quad (33)$$

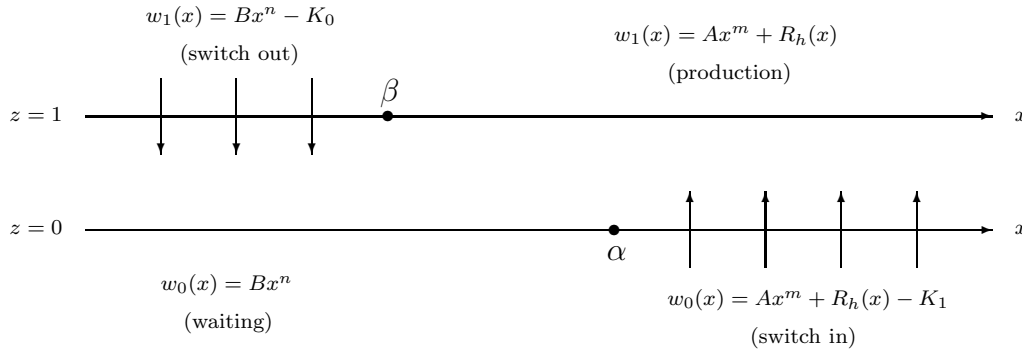


Figure 4. Illustration of the regions determining the optimal strategy in the context of Case II.1

To determine the free-boundary points β , α and the parameters A , B , we once again require that the functions w_1 , w_0 should be C^1 , which yields the expressions

$$A = -\frac{1}{\sigma^2(n-m)} \int_0^\beta s^{-m-1} [h(s) + rK_0] ds, \quad (34)$$

$$B = \frac{1}{\sigma^2(n-m)} \int_\alpha^\infty s^{-n-1} [h(s) - rK_1] ds, \quad (35)$$

and the system of equations

$$m \int_\beta^\alpha s^{-m-1} h(s) ds + rK_0\beta^{-m} + rK_1\alpha^{-m} = 0, \quad (36)$$

$$n \int_\beta^\alpha s^{-n-1} h(s) ds + rK_0\beta^{-n} + rK_1\alpha^{-n} = 0. \quad (37)$$

Lemma 4 *The system of equations (36)–(37) has a unique solution (β, α) such that $0 < \beta < \alpha$ and the functions w_1 , w_0 defined by (32), (33), for $A > 0$, $B > 0$ given by (34), (35), are increasing and satisfy the HJB equation (6)–(7) if and only if*

$$K_0 \leq K \quad \text{and} \quad h(0) < -rK_0.$$

Case II.2 (Figure 5) Abandoning the project if this is in its “open” mode and the state process X takes values below a given threshold δ_\dagger instead of switching it to its “closed” mode is the difference between this case and the previous one.⁵ Accordingly,

$$\mathcal{A}_1 =]0, \delta_\dagger], \quad \mathcal{P} =]\delta_\dagger, \infty[, \quad \mathcal{W} =]0, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{S}_{\text{out}} = \mathcal{A}_0 = \emptyset,$$

and the functions defined by

$$w_1(x) = \begin{cases} -K, & \text{if } x \leq \delta_\dagger \\ Ax^m + R_h(x), & \text{if } x > \delta_\dagger \end{cases} \quad (38)$$

$$\text{and } w_0(x) = \begin{cases} Bx^n, & \text{if } x < \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{cases} \quad (39)$$

should provide a solution to the HJB equation (6)–(7).

⁵We use the notation δ_\dagger rather than the simpler δ because this point will appear in assumptions that we will make in later cases.

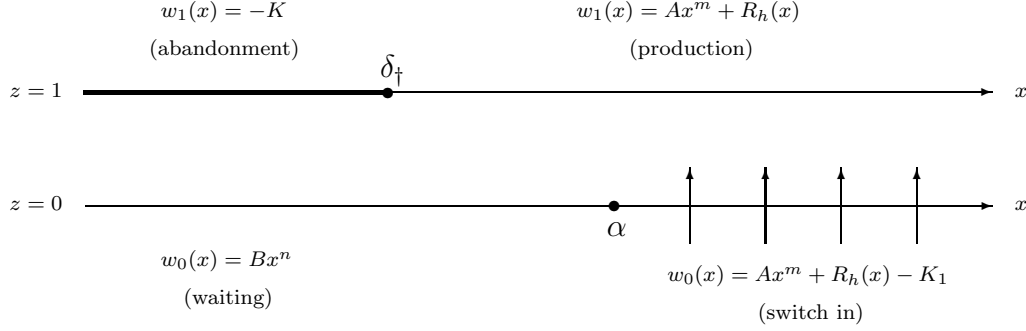


Figure 5. Illustration of the regions determining the optimal strategy in the context of Case II.2

Requiring that w_1, w_0 should be C^1 , we obtain the expressions

$$A = -\frac{1}{\sigma^2(n-m)} \int_0^{\delta_+} s^{-m-1} [h(s) + rK] ds, \quad (40)$$

$$B = \frac{1}{\sigma^2(n-m)} \int_\alpha^\infty s^{-n-1} [h(s) - rK_1] ds, \quad (41)$$

and the system of equations

$$\int_{\delta_+}^\infty s^{-n-1} [h(s) + rK] ds = 0, \quad (42)$$

$$f(\delta, \alpha) := m \int_{\delta_+}^\alpha s^{-m-1} [h(s) - rK_1] ds + r(K_1 + K)\delta_+^{-m} = 0. \quad (43)$$

Lemma 5 *The system of equations (42)–(43) has a unique solution (δ_+, α) such that $0 < \delta_+ < \alpha$ and the functions w_1, w_0 defined by (38), (39), for $A > 0, B > 0$ given by (40), (41), are increasing and satisfy the HJB equation (6)–(7) if and only if*

$$0 \leq K$$

and

$$\left(K < K_0 \text{ and } -rK_0 \leq h(0) < -rK \right) \quad \text{or} \quad \left(K_0^* \leq K_0 \text{ and } h(0) < -rK_0 \right),$$

where $K_0^* \in]K, -r^{-1}h(0)[$, which depends on all problem data except K_0 , is defined by (100) in the proof of the lemma.

Case II.3 (Figure 6) The last case in this group is a hybrid of the previous two. If the investment project is initially in its “open” mode and the initial value x of the process X is

greater than a threshold γ or it is initially in its “closed” mode, then it is optimal to follow the same strategy as in Case II.1, which is determined by two thresholds $\beta < \alpha$ such that $\gamma < \beta$. In this case, the project is sequentially switched from “open” to “closed” and vice versa, and it is never abandoned. On the other hand, if the project is initially in its “open” mode and the initial value x of X is strictly less than γ , then it is optimal to abandon the project as soon as X falls below another threshold $\delta < \gamma$ before hitting γ . Otherwise, it is optimal to switch the project to its “closed” mode if X rises to γ before hitting δ , and then maintain the sequential switching strategy defined by β and α . Accordingly,

$$\begin{aligned}\mathcal{A}_1 &=]0, \delta], \quad \mathcal{P} =]\delta, \gamma[\cup]\beta, \infty[, \quad \mathcal{S}_{\text{out}} = [\gamma, \beta], \\ \mathcal{W} &=]0, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{A}_0 = \emptyset.\end{aligned}$$

In view of (10)–(12) and (21)–(22), we can see that the required solution to the HJB equation (6)–(7) should be given by the functions defined by

$$w_1(x) = \begin{cases} -K, & \text{if } x \leq \delta \\ \Gamma_1 x^m + \Gamma_2 x^n + R_h(x), & \text{if } \delta < x < \gamma \\ Bx^n - K_0, & \text{if } \gamma \leq x \leq \beta \\ Ax^m + R_h(x), & \text{if } x > \beta \end{cases} \quad (44)$$

$$\text{and } w_0(x) = \begin{cases} Bx^n, & \text{if } x < \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{cases}. \quad (45)$$

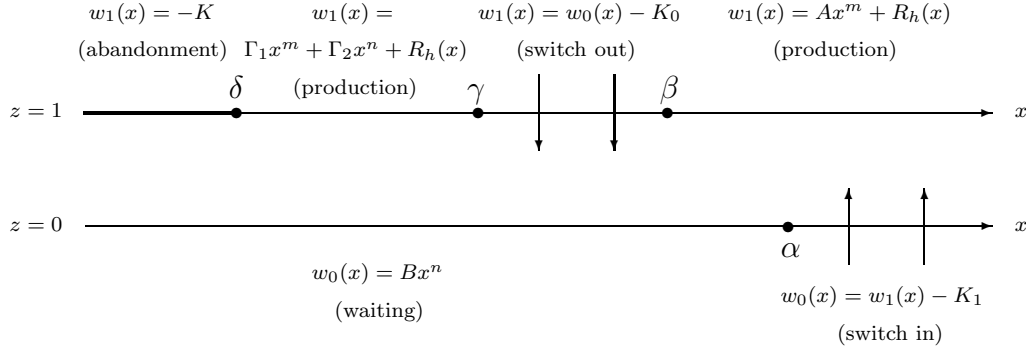


Figure 6. Illustration of the regions determining the optimal strategy in the context of Case II.3

To determine Γ_1 , Γ_2 , A , B , δ , γ , β and α we require that w_1 , w_0 should be C^1 at the free-boundary points δ , γ , β and α . In view of this requirement, we can verify that δ , γ , β and

α should satisfy the equations (36), (37),

$$F_1(\delta, \gamma) := m \int_{\delta}^{\gamma} s^{-m-1} [h(s) + rK_0] ds + r(K - K_0)\delta^{-m} = 0 \quad (46)$$

$$\text{and } F_2(\delta, \gamma) := n \int_{\delta}^{\gamma} s^{-n-1} [h(s) + rK_0] ds + r(K - K_0)\delta^{-n} \\ + n \int_{\beta}^{\infty} s^{-n-1} [h(s) + rK_0] ds = 0, \quad (47)$$

while A , B , Γ_1 and Γ_2 should be given by (34), (35),

$$\Gamma_1 = -\frac{1}{\sigma^2(n-m)} \int_0^{\gamma} s^{-m-1} [h(s) + rK_0] ds \quad (48)$$

$$\text{and } \Gamma_2 = -\frac{1}{\sigma^2(n-m)} \int_{\gamma}^{\beta} s^{-n-1} [h(s) + rK_0] ds. \quad (49)$$

Lemma 6 *The system of equations (36), (37), (46) and (47) has a unique solution $(\delta, \gamma, \beta, \alpha)$ such that $0 < \delta < \gamma < \beta < \alpha$ and the functions w_1 , w_0 defined by (44), (45), for $A > 0$, $B > 0$, $\Gamma_1 > 0$, $\Gamma_2 > 0$ given by (34), (35), (48), (49), are increasing and satisfy the HJB equation (6)–(7) if and only if*

$$0 \leq K, \quad h(0) < -rK_0 \quad \text{and} \quad K < K_0 < K_0^*,$$

where $K_0^* \in]K, -r^{-1}h(0)[$, which depends on all problem data except K_0 , is as in Lemma 5.

We note that the conditions of this result can all hold true only if $h(0) < 0$.

4.3 Group III: the remaining cases

We now consider the remaining cases. These are characterised by the fact that it may be optimal to abandon the investment project when this is in its “closed” mode.

Case III.1 (Figure 7) This case is the modification of Case II.2 (see Figure 5) that arises if abandonment when the project is in its “closed” mode becomes part of the optimal tactics. In this case,

$$\mathcal{A}_1 =]0, \delta_{\dagger}], \quad \mathcal{P} =]\delta_{\dagger}, \infty[, \quad \mathcal{A}_0 =]0, \zeta], \quad \mathcal{W} =]\zeta, \alpha[, \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[\quad \text{and} \quad \mathcal{S}_{\text{out}} = \emptyset,$$

and the functions defined by

$$w_1(x) = \begin{cases} -K, & \text{if } x \leq \delta_{\dagger} \\ Ax^m + R_h(x), & \text{if } x \geq \delta_{\dagger} \end{cases} \quad (50)$$

$$\text{and } w_0(x) = \begin{cases} -K, & \text{if } x \leq \zeta \\ \Delta_1 x^m + \Delta_2 x^n, & \text{if } \zeta \leq x \leq \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{cases} \quad (51)$$

should provide a solution to the HJB equation (6)–(7).

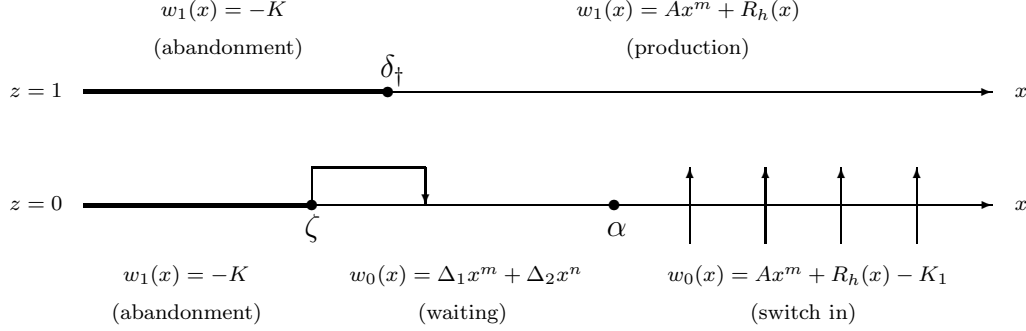


Figure 7. Illustration of the regions determining the optimal strategy in the context of Case III.1 (ζ can be smaller as well as larger than δ_+)

To determine A , Δ_1 , Δ_2 , δ_+ , ζ and α we require that w_1 , w_0 should be C^1 at the free-boundary points δ_+ , ζ and α . In view of this requirement, we can verify that δ_+ , ζ and α should satisfy the system of equations given by (42),

$$G_1(\delta_+, \zeta, \alpha) := m \int_{\delta_+}^{\alpha} s^{-m-1} [h(s) - rK_1] ds + r(K_1 + K)\delta_+^{-m} - rK\zeta^{-m} = 0 \quad (52)$$

$$\text{and } G_2(\delta_+, \zeta, \alpha) := -n \int_{\delta_+}^{\alpha} s^{-n-1} [h(s) - rK_1] ds - r(K_1 + K)\delta_+^{-n} + rK\zeta^{-n} = 0, \quad (53)$$

while, A , Δ_1 and Δ_2 should be given by (40),

$$\Delta_1 = A + \frac{1}{\sigma^2(n-m)} \int_0^{\alpha} s^{-m-1} [h(s) - rK_1] ds = \frac{rK\zeta^{-m}}{\sigma^2 m(n-m)} \quad (54)$$

$$\text{and } \Delta_2 = \frac{1}{\sigma^2(n-m)} \int_{\alpha}^{\infty} s^{-n-1} [h(s) - rK_1] ds = -\frac{rK\zeta^{-n}}{\sigma^2 n(n-m)}. \quad (55)$$

Lemma 7 *The system of equations (42), (52) and (53) has a unique solution $(\delta_+, \zeta, \alpha)$ such that $0 < \delta_+ \wedge \zeta \leq \delta_+ \vee \zeta < \alpha$ and the functions w_1 , w_0 defined by (50), (51), for $A > 0$, $\Delta_1 > 0$, $\Delta_2 > 0$ given by (40), (54), (55), are increasing and satisfy the HJB equation (6)–(7) if and only if*

$$K < 0, \quad h(0) < -rK$$

and

$$\begin{aligned} & \left(-rK_0 \leq h(0) \right) \quad \text{or} \quad \left(h(0) < -rK_0 \text{ and } h(\delta_+) \geq 0 \right) \\ & \quad \text{or} \quad \left(h(0) < -rK_0, \quad h(\delta_+) < 0 \text{ and } K_1 \geq K_1^\dagger \right) \\ & \text{or} \quad \left(h(0) < -rK_0, \quad h(\delta_+) < 0, \quad K_1 < K_1^\dagger \text{ and } K_0 \geq K_0^\dagger \right), \end{aligned}$$

where $K_1^\dagger > 0$ (resp., $K_0^\dagger > 0$), which depends on all of the problem data except K_1 and K_0 (resp., K_0), is given by (126) (resp., (141)) in the proof of the lemma. Furthermore, $\lim_{K_1 \uparrow K_1^\dagger} K_0^\dagger(K_1) = 0$, and the free-boundary points ζ and δ , which do not depend on K_0 , are such that

$$0 < \zeta < \delta \quad \text{if } h(\delta_\dagger) < 0 \text{ and } K_1 < K_1^\dagger, \quad (56)$$

$$0 < \zeta = \delta \quad \text{if } h(\delta_\dagger) < 0 \text{ and } K_1 = K_1^\dagger \quad (57)$$

$$\text{and } 0 < \delta < \zeta \quad \text{if } h(\delta_\dagger) \geq 0 \text{ or } (h(\delta_\dagger) < 0 \text{ and } K_1 > K_1^\dagger). \quad (58)$$

Case III.2 (Figure 8) This case is the modification of Case II.3 that arises when it is optimal to abandon the project when this is in its “closed” mode and the process X takes sufficiently low values. In this case,

$$\begin{aligned} \mathcal{A}_1 &=]0, \delta], \quad \mathcal{P} =]\delta, \gamma[\cup]\beta, \infty[, \quad \mathcal{S}_{\text{out}} = [\gamma, \beta], \\ \mathcal{A}_0 &=]0, \zeta], \quad \mathcal{W} =]\zeta, \alpha[\quad \text{and} \quad \mathcal{S}_{\text{in}} = [\alpha, \infty[, \end{aligned}$$

and the required solution to the HJB equation (6)–(7) should be given by the functions

$$w_1(x) = \begin{cases} -K, & \text{if } x \leq \delta \\ \Gamma_1 x^m + \Gamma_2 x^n + R_h(x), & \text{if } \delta < x < \gamma \\ \Delta_1 x^m + \Delta_2 x^n - K_0, & \text{if } \gamma \leq x \leq \beta \\ Ax^m + R_h(x), & \text{if } x > \beta \end{cases} \quad (59)$$

$$\text{and } w_0(x) = \begin{cases} -K, & \text{if } x \leq \zeta \\ \Delta_1 x^m + \Delta_2 x^n, & \text{if } \zeta < x < \alpha \\ Ax^m + R_h(x) - K_1, & \text{if } x \geq \alpha \end{cases}. \quad (60)$$

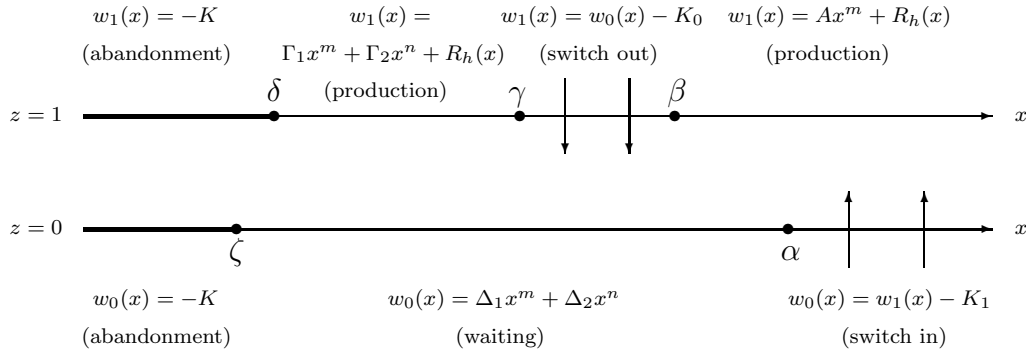


Figure 8. Illustration of the regions determining the optimal strategy in the context of Case III.2

Once again, we specify $\Gamma_1, \Gamma_2, A, \Delta_1, \Delta_2, \zeta, \delta, \gamma, \beta$ and α by requiring that the functions w_1, w_0 should be C^1 . This requirement implies that the free-boundary points $\zeta, \delta, \gamma, \beta$ and α should satisfy the system of equations given by (36), (37),

$$\begin{aligned} G_3(\delta, \gamma, \beta) &:= n \int_{\delta}^{\infty} s^{-n-1} [h(s) + rK] ds - n \int_{\gamma}^{\beta} s^{-n-1} [h(s) + rK_0] ds \\ &= 0, \end{aligned} \quad (61)$$

$$\begin{aligned} G_4(\zeta, \beta) &:= n \int_{\beta}^{\infty} s^{-n-1} [h(s) + rK_0] ds + rK\zeta^{-n} \\ &= 0 \end{aligned} \quad (62)$$

$$\begin{aligned} \text{and } G_5(\zeta, \delta, \gamma) &:= m \int_0^{\gamma} s^{-m-1} [h(s) + rK_0] ds - m \int_0^{\delta} s^{-m-1} [h(s) + rK] ds - rK\zeta^{-m} \\ &= 0, \end{aligned} \quad (63)$$

while the constants $\Gamma_1, \Gamma_2, A, \Delta_1, \Delta_2$ should be given by

$$\Gamma_1 = -\frac{1}{\sigma^2(n-m)} \int_0^{\delta} s^{-m-1} [h(s) + rK] ds, \quad (64)$$

$$\Gamma_2 = -\frac{1}{\sigma^2(n-m)} \int_{\delta}^{\infty} s^{-n-1} [h(s) + rK] ds, \quad (65)$$

$$\Delta_1 = \frac{rK\zeta^{-m}}{\sigma^2 m(n-m)}, \quad \Delta_2 = -\frac{rK\zeta^{-n}}{\sigma^2 n(n-m)} \quad (66)$$

$$\text{and } A = \Delta_1 - \frac{1}{\sigma^2(n-m)} \int_0^{\alpha} s^{-m-1} [h(s) - rK_1] ds. \quad (67)$$

Lemma 8 *The system of equations (36), (37), (61), (62), (63) has a unique solution $(\delta, \gamma, \beta, \alpha)$ such that $0 < \delta < \gamma < \beta < \alpha$ and the functions w_1, w_0 defined by (59), (60), for $\Gamma_1 > 0, \Gamma_2 > 0, A > 0, \Delta_1 > 0, \Delta_2 > 0$ given by (64)–(67), are increasing and satisfy the HJB equation (6)–(7) if and only if*

$$K < 0, \quad h(0) < -rK_0, \quad h(\delta_{\dagger}) < 0, \quad K_1 < K_1^{\dagger} \quad \text{and} \quad K_0 < K_0^{\dagger},$$

where $\delta_{\dagger} > 0$ is the unique solution to (42), and $K_1^{\dagger} > 0$ (resp., $K_0^{\dagger} > 0$), which depends on all problem data except K_1 and K_0 (resp., K_0) is as in Lemma 7.

4.4 The main result

The following table summarises the conditions on the problem data that determine the optimality of each of the cases that we have studied in Sections 4.1–4.3. An inspection of the table reveals that these mutually exclusive conditions exhaust the whole range of possible problem data. Therefore, Lemmas 1–8 provide a complete solution to the HJB equation (6)–(7).

Conditions on $K_1 > 0, K_0 > 0, K \in \mathbb{R}$ and h		Case	w_1, w_0
$0 \leq K$	$rK_1 \leq h(0)$	I.1, Lemma 1	(23), (24)
	$\max\{-rK_0, -rK\} \leq h(0) < rK_1$	I.2, Lemma 2	(23), (25)
	$K_0 \leq K$ and $h(0) < -rK_0$	II.1, Lemma 4	(33), (32)
	$K < K_0$ and $-rK_0 \leq h(0) < -rK$	II.2, Lemma 5	(38), (39)
	$K < K_0^* \leq K_0$ and $h(0) < -rK_0$	II.2, Lemma 5	(38), (39)
	$K < K_0 < K_0^*$ and $h(0) < -rK_0$	II.3, Lemma 6	(44), (45)
$K < 0$	$rK_1 - rK \leq h(0)$	I.1, Lemma 1	(23), (24)
	$-rK \leq h(0) < rK_1 - rK$	I.3, Lemma 3	(23), (28)
	$-rK_0 \leq h(0) < -rK$	III.1, Lemma 7	(50), (51)
	$h(0) < -rK_0$ and $h(\delta_+) \geq 0$ or $(h(\delta_+) < 0$ and $K_1 \geq K_1^\dagger)$ or $(h(\delta_+) < 0, K_1 < K_1^\dagger$ and $K_0 \geq K_0^\dagger)$	III.1, Lemma 7	(50), (51)
	$h(0) < -rK_0,$ $h(\delta_+) < 0, K_1 < K_1^\dagger$ and $K_0 < K_0^\dagger$	III.2, Lemma 8	(59), (60)

Theorem 9 *Consider the stochastic optimal control problem formulated in Section 2 and suppose that Assumption 1 holds true. The value function v is given by (5), where w_1, w_0 are as in Lemmas 1-8. In each of the possible cases arising, the optimal strategy (Z°, τ°) is as discussed in the proof below.*

Proof. Given any initial condition $(z, x) \in \{0, 1\} \times]0, \infty[$ and any strategy $(Z, \tau) \in \Pi_z$, the monotone convergence theorem and (4) in Assumption 1 imply that $\lim_{m \rightarrow \infty} J_{z,x}(Z, \tau \wedge T_m) = J_{z,x}(Z, \tau)$ for every sequence of times (T_m) such that $T_m \rightarrow \infty$. By construction, there exists a constant $C > 0$ such that

$$|w(z, x)| \leq C(1 + |R_h(x)|) \quad \text{and} \quad |w_x(z, x)| \leq C(1 + |R'_h(x)|) \quad \text{for all } x > 0,$$

where $w(z, x) = zw_1(x) + (1 - z)w_0(x)$. These estimates, (18) and (19) imply that

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[e^{-rT} |w(Z_T, X_T)| \right] = 0,$$

and that the process M defined by

$$M_T = \int_0^T e^{-rt} X_t w_x(Z_t, X_t) dW_t$$

is a square integrable martingale for every switching strategy $Z \in \mathcal{Z}$. Furthermore, w_1, w_0 are C^1 as well as C^2 outside a finite set, and they satisfy the HJB equation (6)–(7) in the classical sense. In view of these observations, we can see that Theorem 1 in Zervos [29] implies that $w = v$ as long as there exists an optimal strategy (Z°, τ°) , namely, a switching strategy $Z^\circ \in \mathcal{Z}$ such that

$$\begin{aligned} \sigma^2 X_t^2 w_{xx}(Z_t^\circ, X_t) + b X_t w_x(Z_t^\circ, X_t) - r w(Z_t^\circ, X_t) + Z_t^\circ h(X_t) &= 0, \\ [w(1, X_t) - w(0, X_t) - K_1](\Delta Z_t^\circ)^+ &= 0 \\ \text{and } [w(0, X_t) - w(1, X_t) - K_0](\Delta Z_t^\circ)^- &= 0, \end{aligned}$$

for all $t \leq \tau^\circ$, where

$$\tau^\circ = \inf \{t \geq 0 \mid w(Z_t^\circ, X_t) = -K\}.$$

Such a switching strategy is constructed in Duckworth and Zervos [8, Theorem 5] and Zervos [29, Theorem 1] for Cases I.1, I.2, II.1, II.2 and II.3. For the remaining cases, it can be constructed using similar arguments. \square

Appendix I: auxiliary results

We first prove the following results that we will use to streamline the proofs of Lemmas 1-8 in the next appendix.

Lemma 10 *Suppose that the function h satisfies the requirements of Assumption 1. Given any constants $\nu \geq 0$ and L ,*

$$\lim_{x \rightarrow \infty} \int_\nu^x s^{-m-1} [h(s) + L] ds = \infty. \quad (68)$$

Also, given any $\nu \in]0, \infty]$ and L such that $h(0) + L < 0$,

$$\lim_{x \downarrow 0} \int_x^\nu s^{-n-1} [h(s) + L] ds = -\infty. \quad (69)$$

Proof. Since h is increasing and $\lim_{x \rightarrow \infty} h(x) = \infty$, there exist constants $x_1 > \nu$, $M > 0$ such that $h(x) + L > M$ for all $x \geq x_1$. Therefore,

$$\lim_{x \rightarrow \infty} \int_\nu^x s^{-m-1} [h(s) + L] ds \geq \lim_{x \rightarrow \infty} \left[\int_\nu^{x_1} s^{-m-1} [h(s) + L] ds + \frac{M}{m} x_1^{-m} - \frac{M}{m} x^{-m} \right] = \infty$$

because $m < 0$. The assumption $h(0) + L < 0$ implies that there exist constants $x_2 > 0$ and $\varepsilon > 0$ such that $h(x) + L \leq -\varepsilon$ for all $x \leq x_2$. It follows that

$$\lim_{x \downarrow 0} \int_x^\nu s^{-n-1} [h(s) + L] ds \leq \lim_{x \downarrow 0} \left[\int_{x_2}^\nu s^{-n-1} [h(s) + L] ds + \frac{\varepsilon}{n} x_2^{-n} - \frac{\varepsilon}{n} x^{-n} \right] = -\infty$$

because $n > 0$. □

We have assumed that h is increasing and right-continuous rather than strictly increasing and continuous. Therefore, the statements as well as the proofs of the following two results have to take into account carefully the possible jumps or intervals of constancy of h .

Lemma 11 *Suppose that the function h satisfies the requirements of Assumption 1. Fix any constants ν and L such that $h(0) + L < 0$ and*

$$\nu > \inf \{x > 0 \mid h(x) + L > 0\} \geq \inf \{x > 0 \mid h(x) + L \geq 0\} =: \underline{\nu},$$

and consider the function $q :]0, \nu] \rightarrow \mathbb{R}$ defined by

$$q(x) = \frac{mx^{m-1}}{\sigma^2(n-m)} \int_x^\nu s^{-m-1} [h(s) + L] ds - \frac{nx^{n-1}}{\sigma^2(n-m)} \int_x^\nu s^{-n-1} [h(s) + L] ds.$$

There exists a unique $\hat{x} \in [0, \underline{\nu}[$ such that

$$q(x) \begin{cases} > 0 & \text{for all } x \in]0, \hat{x}[, \text{ if } \hat{x} > 0 \\ < 0 & \text{for all } x \in]\hat{x}, \nu[\end{cases}.$$

Proof. We define $p(x) = \sigma^2 x^{-m+1} q(x)$, we note that

$$p(\nu) = 0 \quad \text{and} \quad p(x) < 0 \quad \text{for all } x \in [\underline{\nu}, \nu[, \tag{70}$$

and we use the integration by parts formula to calculate

$$\begin{aligned} p'(x) &= x^{n-m-1} \left(x^{-n} [h(x) + L] - n \int_x^\nu s^{-n-1} [h(s) + L] ds \right) \\ &= x^{n-m-1} \left(\nu^{-n} [h(\nu) + L] - \int_{[x, \nu]} s^{-n} dh(s) \right) =: x^{n-m-1} u(x), \end{aligned}$$

where we have taken into account that h is right-continuous. The function u is increasing because h is an increasing function. Combining this observation with the assumption that $h(\nu) + L > 0$, we can see that, if we define

$$\bar{x} := \inf \{x \in]0, \nu] \mid u(x) \geq 0\} \leq \inf \{x \in]0, \nu] \mid u(x) > 0\} =: \bar{x},$$

then $\underline{x}, \bar{x} \in [0, \nu]$ and

$$p'(x) \begin{cases} < 0 & \text{for all } x \in]0, \underline{x}[, \text{ if } \underline{x} > 0 \\ = 0 & \text{for all } x \in [\underline{x}, \bar{x}] , \text{ if } \bar{x} > 0 \\ > 0 & \text{for all } x \in]\bar{x}, \nu[, \text{ if } \bar{x} < \nu \end{cases}.$$

These inequalities and (70) imply that $\bar{x} < \nu$. Furthermore, if $\hat{x} = \inf\{x \in]0, \nu] \mid p(x) < 0\}$, then $\hat{x} > 0$ if and only if $\lim_{x \downarrow 0} p(x) > 0$, and, if $\hat{x} > 0$, then $p'(\hat{x}) < 0$. The required conclusions follow from these observations and the fact that $p(x)$ and $q(x)$ have the same sign. \square

Lemma 12 *Suppose that the function h satisfies the requirements of Assumption 1. Fix any constants $\nu > 0$ and L such that $h(\nu) + L < 0$, and consider the function $q :]\nu, \infty[\rightarrow \mathbb{R}$ defined by*

$$q(x) = \frac{mx^{m-1}}{\sigma^2(n-m)} \int_{\nu}^x s^{-m-1} [h(s) + L] ds - \frac{nx^{n-1}}{\sigma^2(n-m)} \int_{\nu}^x s^{-n-1} [h(s) + L] ds.$$

There exists a unique $\hat{x} \in]\bar{\nu}, \infty]$ such that

$$q(x) \begin{cases} > 0 & \text{for all } x \in]\nu, \hat{x}[\\ < 0 & \text{for all } x > \hat{x}, \text{ if } \hat{x} < \infty \end{cases},$$

where

$$\bar{\nu} := \inf\{x > 0 \mid h(x) + L > 0\} \geq \inf\{x > 0 \mid h(x) + L \geq 0\} =: \underline{\nu} > \nu.$$

Proof. We define $p(x) = \sigma^2 x^{-m+1} q(x)$ and we note that the right continuity of h implies that $\underline{\nu} > \nu$. A simple inspection of the definition of p reveals that

$$p(\nu) = 0 \quad \text{and} \quad p(x) > 0 \text{ for all } x \in]\nu, \underline{\nu}]. \quad (71)$$

Using the integration by parts formula, we calculate

$$\begin{aligned} p'(x) &= -x^{n-m-1} \left(x^{-n} [h(x) + L] + n \int_{\nu}^x s^{-n-1} [h(s) + L] ds \right) \\ &= -x^{n-m-1} \left(\nu^{-n} [h(\nu) + L] + \int_{[\nu, x]} s^{-n} dh(s) \right) =: -x^{n-m-1} u(x), \end{aligned}$$

where we have taken into account that h is right-continuous. The function u is increasing because h is an increasing function. In view of this observation and the assumption that $h(\nu) + L < 0$, we can see that, if we define

$$\underline{x} := \sup\{x \geq \nu \mid u(x) < 0\} \leq \sup\{x \geq \nu \mid u(x) \leq 0\} =: \bar{x},$$

then $\underline{x}, \bar{x} \in [\nu, \infty]$ and

$$p'(x) \begin{cases} > 0 & \text{for all } x \in]\nu, \underline{x}[, \text{ if } \underline{x} > \nu \\ = 0 & \text{for all } x \in [\underline{x}, \bar{x}] , \text{ if } \bar{x} > \nu \\ < 0 & \text{for all } x \in]\bar{x}, \infty[, \text{ if } \bar{x} < \infty \end{cases}.$$

These inequalities and (71) imply that $\underline{x} > \nu$. If we define $\hat{x} = \sup\{x \geq \nu \mid p(x) > 0\}$, then $\hat{x} < \infty$ if and only if $\lim_{x \rightarrow \infty} p(x) < 0$, and, if $\hat{x} < \infty$, then $p'(\hat{x}) < 0$. The required conclusions follow from these observations and the fact that $p(x)$ and $q(x)$ have the same sign. \square

Lemma 13 *Suppose that the function h satisfies the requirements of Assumption 1. The function $f :]0, \infty[\rightarrow \mathbb{R}$ defined by $f(x) = x^{-m+1} R'_h(x)$ is strictly increasing.*

Proof. Using the expression (15) for R'_h and the integration by parts formula, we calculate

$$f'(x) = \frac{1}{\sigma^2} x^{n-m-1} \left(-x^{-n} h(x) + n \int_x^\infty s^{-n-1} h(s) ds \right) = \frac{1}{\sigma^2} x^{n-m-1} \int_x^\infty s^{-n} dh(s),$$

and the claim follows because h is increasing and $\lim_{x \rightarrow \infty} h(x) = \infty$. \square

We will also need the following simple real analysis result.

Lemma 14 *Given points $0 < z_1 < z_2$ and $\kappa \in \mathbb{R}$, if $f : [z_1, z_2] \rightarrow \mathbb{R}$ is any right-continuous increasing function that is not identically 0 and is such that*

$$\int_{z_1}^{z_2} s^\kappa f(s) ds = 0, \tag{72}$$

then

$$\int_{z_1}^{z_2} s^\mu f(s) ds < 0 \quad \text{for all } \mu < \kappa. \tag{73}$$

Proof. We define $y = \inf\{x \in [z_1, z_2] \mid f(x) \geq 0\}$, and we note that (72) and the fact that f is increasing and not identically 0 imply that $y \in]z_1, z_2[$, $f(x) < 0$ for all $x \in [z_1, y[$ and $f(x) \geq 0$ for all $x \in [y, z_2]$. In view of these observations, we can see that, given any $\mu < \kappa$,

$$0 = \int_{z_1}^{z_2} s^{\kappa-\mu} s^\mu f(s) ds > y^{\kappa-\mu} \int_{z_1}^y s^\mu f(s) ds + y^{\kappa-\mu} \int_y^{z_2} s^\mu f(s) ds,$$

and (73) follows. \square

Appendix II: proof of Lemmas 1–8

In each of the proofs, we mark with bold the first occurrence of each of the conditions determining the optimality of the case.

Proof of Lemmas 1, 2 and 4. The functions w_1, w_0 defined by (23), (24) satisfy the HJB equation

$$\max \{ \sigma^2 x^2 w_1''(x) + bxw_1'(x) - rw_1(x) + h(x), w_0(x) - K_0 - w_1(x) \} = 0, \quad (74)$$

$$\max \{ \sigma^2 x^2 w_0''(x) + bxw_0'(x) - rw_0(x), w_1(x) - K_1 - w_0(x) \} = 0 \quad (75)$$

if and only if $\mathbf{rK_1} \leq \mathbf{h(0)}$ (see Duckworth and Zervos [8, Lemma 2]). These functions will satisfy the HJB equation (6)–(7) if and only if

$$-w_1(x) - K \leq 0 \quad \text{and} \quad -w_0(x) - K \leq 0 \quad \text{for all } x > 0. \quad (76)$$

In view of (17), the first of these inequalities is true if and only if $-rK \leq h(0)$, while the second one is true if and only if $\mathbf{rK_1} - \mathbf{rK} \leq \mathbf{h(0)}$, and Lemma 1 follows because $K_1 > 0$.

Equation (27) has a unique solution $\alpha > 0$ and the functions w_1, w_0 defined by (23), (25) and (26) are increasing and satisfy the HJB equation (74)–(75) if and only if $-\mathbf{rK_0} \leq \mathbf{h(0)} < \mathbf{rK_1}$ (see Duckworth and Zervos [8, Lemma 3]). These functions will satisfy the HJB equation (6)–(7) if only if they satisfy (76). The first of these inequalities holds true if and only if $-\mathbf{rK} \leq \mathbf{h(0)}$, while the second one is true if and only if $\mathbf{K} \geq \mathbf{0}$ because $\lim_{x \downarrow 0} w_0(x) = 0$ and w_0 is increasing, and Lemma 2 follows.

The system of equations (36)–(37) has a unique solution (α, β) such that $0 < \beta < \alpha$ if and only if $\mathbf{h(0)} < -\mathbf{rK_0}$, in which case, the functions w_1, w_0 defined by (33)–(35) are increasing and satisfy the HJB equation (74)–(75) (see Duckworth and Zervos [8, Lemma 3]). Furthermore, the solution (α, β) is such that

$$\beta < \inf \{ x > 0 \mid h(x) + rK_0 \geq 0 \} \quad \text{and} \quad \alpha > \sup \{ x > 0 \mid h(x) - rK_1 \leq 0 \}. \quad (77)$$

The functions w_1, w_0 will satisfy the HJB equation (6)–(7) if and only if they satisfy (76). Both of these inequalities will be true if and only if $\mathbf{K} \geq \mathbf{K_0}$ because w_1, w_0 are increasing and $\lim_{x \downarrow 0} w_1(x) = \lim_{x \downarrow 0} w_0(x) - K_0 = -K_0$, and Lemma 4 follows. \square

Proof of Lemma 3. In view of the monotonicity of h , a simple inspection of (29)–(30) reveals that this system of equations has no solution if $K = 0$. On the other hand, the functions w_1, w_0 defined by (23), (28) can satisfy the HJB equation (6)–(7) only if $\sigma^2 x^2 w_0''(x) + bxw_0'(x) - rw_0(x) = rK \leq 0$ for all $x < \zeta$. We therefore assume that $\mathbf{K} < \mathbf{0}$ in what follows.

To establish conditions under which the system of equations (29)–(30) has a unique solution (ζ, α) such that $0 < \zeta < \alpha$ when $K < 0$, we define

$$\begin{aligned} \underline{\alpha} &:= \inf \{ x > 0 \mid h(x) - rK_1 \geq 0 \} \leq \inf \{ x > 0 \mid h(x) - rK_1 > 0 \} =: \bar{\alpha} \\ \text{and} \quad \underline{\zeta} &:= \inf \{ x > 0 \mid h(x) + rK - rK_1 \geq 0 \} \geq \bar{\alpha}, \end{aligned} \quad (78)$$

and we note that $\bar{\alpha} = \zeta$ if and only if $\zeta = 0$. If $\zeta > 0$, then the assumption that h is increasing and (68) in Lemma 10 imply that there exists a unique $\hat{\zeta} > \zeta$ such that

$$f_1(\zeta, \zeta) = m \int_0^\zeta s^{-m-1} [h(s) + rK - rK_1] ds \begin{cases} > 0, & \text{if } \zeta \in]0, \hat{\zeta}[\\ < 0, & \text{if } \zeta \in]\hat{\zeta}, \infty[\end{cases}.$$

On the other hand,

$$\text{if } \zeta = \bar{\alpha} = 0 \text{ then } f_1(\zeta, \zeta) < 0 \text{ for all } \zeta > 0.$$

Furthermore, the calculation

$$\frac{\partial f_1(\zeta, \alpha)}{\partial \alpha} = m\alpha^{-m-1} [h(\alpha) - rK_1]$$

implies that (I) $f_1(\zeta, \cdot)$ is strictly increasing in $] \zeta, \alpha[$ and constant in $[\alpha, \bar{\alpha}]$, if $\bar{\alpha} > 0$ and $\zeta < \bar{\alpha}$, and (II) $f_1(\zeta, \cdot)$ is strictly decreasing in $] \zeta \vee \bar{\alpha}, \infty[$. Combining these observations with the fact that $\lim_{\alpha \rightarrow \infty} f_1(\zeta, \alpha) = -\infty$, which follows from (68) in Lemma 10, we can see that, given $\zeta > 0$, there exists a unique $\alpha > \zeta$ such that $f_1(\zeta, \alpha) = 0$ if and only if

$$\zeta > 0 \quad \Leftrightarrow \quad \mathbf{h(0) + rK - rK_1} < \mathbf{0} \quad (79)$$

and $\zeta \in]0, \hat{\zeta}[$. It follows that, if the inequalities in (79) hold true, then there exists a unique function $\ell :]0, \hat{\zeta}[$ such that

$$f_1(\zeta, \ell(\zeta)) = 0 \text{ for all } \zeta \in]0, \hat{\zeta}[\quad \text{and} \quad \lim_{\zeta \uparrow \hat{\zeta}} \ell(\zeta) = \hat{\zeta}.$$

Furthermore, differentiating the identity $f_1(\zeta, \ell(\zeta)) = 0$ with respect to ζ , we obtain

$$\ell'(\zeta) = -\frac{\zeta^{-m-1} rK}{\ell^{-m-1}(\zeta) [h(\ell(\zeta)) - rK_1]} > 0, \quad (80)$$

the inequality following because $K < 0$ and $\ell(\zeta) > \hat{\zeta} > \bar{\alpha}$.

In the presence of (79), we will show that the system of equations (29)–(30) has a unique solution (ζ, α) such that $0 < \zeta < \alpha$ if we prove that the equation $f_2(\zeta, \ell(\zeta)) = 0$ has a unique solution $\zeta \in]0, \hat{\zeta}[$. To this end, we note that

$$\lim_{\zeta \uparrow \hat{\zeta}} f_2(\zeta, \ell(\zeta)) = f_2(\hat{\zeta}, \hat{\zeta}) = n \int_{\hat{\zeta}}^{\infty} s^{-n-1} [h(s) + rK - rK_1] ds > 0$$

and

$$\lim_{\zeta \downarrow 0} f_2(\zeta, \ell(\zeta)) < \lim_{\zeta \downarrow 0} \left(n \int_{\hat{\zeta}}^{\infty} s^{-n-1} [h(s) + rK - rK_1] ds + rK \zeta^{-n} \right) = -\infty.$$

Combining these calculations with

$$\begin{aligned} \frac{df_2(\zeta, \ell(\zeta))}{d\zeta} &= -n\ell^{-n-1}(\zeta)[h(\ell(\zeta)) - rK_1]\ell'(\zeta) - nrk\zeta^{-n-1} \\ &\stackrel{(80)}{=} -nrK\zeta^{-m-1}[\ell^{m-n}(\zeta) - \zeta^{m-n}] > 0, \end{aligned}$$

we can see that the equation $f_2(\zeta, \ell(\zeta)) = 0$ has a unique solution $\zeta \in]0, \hat{\zeta}[$, as required.

The C^1 functions w_1, w_0 defined by (23), (28) are increasing because $w_1'(x) = R_h'(x) \stackrel{(16)}{\geq} 0$ for all $x > 0$, $w_0'(x) = 0$ for all $x \in]0, \zeta]$, $w_0'(x) = R_h'(x) \geq 0$ for all $x \geq \alpha$, and

$$\begin{aligned} w_0'(x) &= m\Delta_1 x^{m-1} + n\Delta_2 x^{n-1} \\ &\stackrel{(31)}{=} -\frac{rK}{\sigma^2(n-m)x} \left[\left(\frac{x}{\zeta}\right)^n - \left(\frac{x}{\zeta}\right)^m \right] > 0 \quad \text{for all } x \in]\zeta, \alpha[. \end{aligned}$$

To show that these increasing functions provide a solution to the HJB equation (6)–(7), we still need to prove that

$$\sigma^2 x^2 w_0'' + bxw_0'(x) - rw_0(x) \leq 0 \quad \text{for all } x > \alpha, \quad (81)$$

$$w_0(x) - w_1(x) - K_0 \leq 0 \quad \text{for all } x > 0, \quad (82)$$

$$w_1(x) - w_0(x) - K_1 \leq 0 \quad \text{for all } x \leq \alpha, \quad (83)$$

$$\text{and} \quad -w_1(x) - K \leq 0 \quad \text{for all } x \geq 0. \quad (84)$$

The inequality (81) is equivalent to $h(x) - rK_1 \geq 0$ for all $x > \alpha$, which is true thanks to the fact that $\alpha > \bar{\alpha}$, where $\bar{\alpha}$ is defined at the beginning of the proof, and the assumption that h is increasing. The inequality (82) for $x \geq \alpha$ is equivalent to $K_1 + K_0 \geq 0$, which is true by assumption. In view of (16)–(17), the inequality (84) is equivalent to $-rK \leq h(0)$. Similarly, the inequality (82) for $x \leq \zeta$ is equivalent to $-rK - rK_0 \leq h(0)$, which is implied by $-rK \leq h(0)$. The inequality (83) for $x < \zeta$ is equivalent to $w_1(x) + K - K_1 \leq 0$ and will follow immediately once we have established (83) for $x \in [\zeta, \alpha]$ because w_1 is increasing.

To establish (82)–(83) for $x \in [\zeta, \alpha]$, and complete the proof, we note that these inequalities are equivalent to

$$-K_1 - K_0 \leq g_1(x) \leq 0 \quad \text{for all } x \in [\zeta, \alpha], \quad (85)$$

where $g_1(x) = w_0(x) - w_1(x) - K_0$. Using (23), (28) and (31) we calculate

$$g_1(x) = \frac{x^m}{\sigma^2(n-m)} \int_x^\alpha s^{-m-1} [h(s) - rK_1] ds - \frac{x^n}{\sigma^2(n-m)} \int_x^\alpha s^{-n-1} [h(s) - rK_1] ds - K_1 - K_0 \quad (86)$$

$$\text{and } g_1'(x) = \frac{mx^{m-1}}{\sigma^2(n-m)} \int_x^\alpha s^{-m-1} [h(s) - rK_1] ds - \frac{nx^{n-1}}{\sigma^2(n-m)} \int_x^\alpha s^{-n-1} [h(s) - rK_1] ds. \quad (87)$$

These expressions, the fact that (82) holds true for all $x \leq \zeta$, and the C^1 continuity of w_1 , w_0 at ζ imply that

$$g_1(\zeta) \leq 0, \quad g_1'(\zeta) = -w_1'(\zeta) \leq 0, \quad g_1(\alpha) = -K_1 - K_0 < 0 \quad \text{and} \quad g_1'(\alpha) = 0.$$

Recalling that $\alpha > \bar{\alpha}$, where $\bar{\alpha}$ is defined at the beginning of the proof, we combine the inequalities $g_1'(\zeta) \leq 0$ and $g_1'(\alpha) = 0$ with Lemma 11 for $\nu = \alpha$, $L = -rK_1$ and $q = g_1'$, to see that $g_1'(x) < 0$ for all $x \in]\zeta, \alpha[$. It follows that $g_1(x)$ decreases from $g_1(\zeta) \leq 0$ to $g_1(\alpha) = -K_1 - K_0$ as x increases from ζ to α , and (85) follows. \square

Proof of Lemma 5. Using (69) in Lemma 10 and the assumptions that h is increasing and $\lim_{x \rightarrow \infty} h(x) = \infty$, we can see that equation (42) has a unique solution $\delta_{\dagger} > 0$ if and only if $h(0) + rK < 0$. Furthermore, the solution δ_{\dagger} is such that

$$h(x) + rK < 0 \quad \text{for all } x \leq \delta_{\dagger}. \quad (88)$$

Before addressing the solvability of (43), we note that the functions w_1 , w_0 defined by (38), (39) can satisfy the HJB equation (6)–(7) only if $w_0(x) = Bx^n \geq -K$ for all $x \leq \alpha$. This inequality cannot be true for x arbitrarily close to 0 if $-K > 0$. Therefore, we assume in what follows that

$$K \geq 0 \quad \Rightarrow \quad K + K_1 > 0, \quad (89)$$

the implication following because $K_1 > 0$.

To show that equation (43) has a unique solution $\alpha > \delta_{\dagger}$, we define

$$\delta_{\dagger} < \underline{\alpha} := \inf\{x > 0 \mid h(x) - rK_1 \geq 0\} \leq \inf\{x > 0 \mid h(x) - rK_1 > 0\} =: \bar{\alpha}. \quad (90)$$

Here, the first inequality follows because h is right-continuous, and (88)–(89) imply that $h(x) - rK_1 < h(x) + rK \leq h(\delta_{\dagger}) + rK < 0$ for all $x \leq \delta_{\dagger}$. In view of the calculation

$$\frac{\partial f(\delta_{\dagger}, \alpha)}{\partial \alpha} = m\alpha^{-m-1} [h(\alpha) - rK_1],$$

we can see that $f(\delta_{\dagger}, \cdot)$ is strictly increasing in $]\delta_{\dagger}, \alpha[$ and strictly decreasing in $]\bar{\alpha}, \infty[$. Combining this observation with the calculation $f(\delta_{\dagger}, \delta_{\dagger}) = r(K + K_1)\delta_{\dagger}^{-m} > 0$ and the fact that $\lim_{\alpha \rightarrow \infty} f(\delta_{\dagger}, \alpha) = -\infty$, which follows from (68) in Lemma 10, we can see that equation (43) has a unique solution $\alpha > \delta_{\dagger}$. Furthermore, this solution satisfies

$$\alpha > \bar{\alpha} := \inf\{x > 0 \mid h(x) - rK_1 > 0\}. \quad (91)$$

In view of (88) and (91), a simple inspection of the expressions (40) and (41) reveals that $A, B > 0$. Also, the expression (15) for R'_h and the fact that δ_{\dagger} satisfies equation (42) imply that $mA = -\delta_{\dagger}^{-m+1}R'_h(\delta_{\dagger})$. Therefore,

$$R'_h(x) + mA x^{m-1} = x^{m-1} \left[x^{1-m} R'_h(x) - \delta_{\dagger}^{1-m} R'_h(\delta_{\dagger}) \right] > 0, \quad \text{for all } x > \delta_{\dagger},$$

the inequality following from Lemma 13. Using these results, it is straightforward to verify that the functions w_1, w_0 defined by (38), (39) are both increasing.

To complete the proof, we need to derive additional conditions under which the functions w_1, w_0 are indeed solutions to the HJB equation (6)–(7). In view of our analysis thus far, this amounts to establishing the inequalities

$$\sigma^2 x^2 w_1''(x) + b x w_1'(x) - r w_1(x) + h(x) \leq 0 \quad \text{for all } x < \delta_{\dagger}, \quad (92)$$

$$\sigma^2 x^2 w_0''(x) + b x w_0'(x) - r w_0(x) \leq 0 \quad \text{for all } x > \alpha, \quad (93)$$

$$w_0(x) - w_1(x) - K_0 \leq 0 \quad \text{for all } x > 0, \quad (94)$$

$$w_1(x) - w_0(x) - K_1 \leq 0 \quad \text{for all } x \leq \alpha, \quad (95)$$

$$-w_1(x) - K \leq 0 \quad \text{for all } x \geq \delta_{\dagger}, \quad (96)$$

$$\text{and} \quad -w_0(x) - K \leq 0 \quad \text{for all } x > 0. \quad (97)$$

The inequality (92) is equivalent to $h(x) + rK \leq 0$ for all $x < \delta_{\dagger}$, which is true thanks to (88). Similarly, (93) follows from (91). The inequality (94) for $x \leq \delta_{\dagger}$ is equivalent to $Bx^n + K - K_0 \leq 0$, which can be true only if $\mathbf{K} < \mathbf{K}_0$, and will follow immediately once we establish (94) for $x \in [\delta_{\dagger}, \alpha]$ because $x \mapsto Bx^n$ is strictly increasing. Also, (94) for $x \geq \alpha$ is equivalent to $K_0 + K_1 > 0$, which is true by assumption. For $x \leq \delta_{\dagger}$, the inequality (95) is equivalent to $Bx^n + K + K_1 \geq 0$, which follows from (89) and the fact that $B > 0$. Furthermore, (96) and (97) hold true because w_1, w_0 are increasing, $w_1(\delta_{\dagger}) = -K$, $\lim_{x \downarrow 0} w_0(x) = 0$ and $K \geq 0$.

The inequalities (94) and (95) for $x \in [\delta_{\dagger}, \alpha]$ are equivalent to

$$-K_1 - K_0 \leq g_1(x) \leq 0 \quad \text{for all } x \in [\delta_{\dagger}, \alpha], \quad (98)$$

where $g_1(x) = w_0(x) - w_1(x) - K_0$. Using (38)–(41) and (43), we can verify that g_1 and g'_1 admit the expressions given by (86) and (87). These expressions, the inequality (95) for $x \leq \delta_{\dagger}$, which we have established above, and the C^1 continuity of w_1, w_0 at δ_{\dagger} imply that

$$g_1(\delta_{\dagger}) \geq -K_1 - K_0, \quad g'_1(\delta_{\dagger}) = nB\delta_{\dagger}^{-n-1} > 0, \quad g_1(\alpha) = -K_1 - K_0 \quad \text{and} \quad g'_1(\alpha) = 0.$$

In view of these results, (91) and Lemma 11 for $\nu = \alpha$, $L = -rK_1$ and $q = g'_1$, we can see that there exists $\hat{x} \in]\delta_+, \alpha[$, where α is defined by (90), such that

$$g'_1(x) \begin{cases} > 0 & \text{for all } x \in [\delta_+, \hat{x}[\\ < 0 & \text{for all } x \in]\hat{x}, \alpha[\end{cases}.$$

It follows that g_1 has a unique maximum in $[\delta_+, \alpha]$ and (98) holds true if and only if $g_1(\hat{x}) \leq 0$. Using the expressions (86), (87) of g_1 , g'_1 , equation (43), and the identity $\sigma^2 mn = -r$, we calculate

$$\begin{aligned} g_1(\hat{x}) &= -\frac{m\hat{x}^m}{r} \int_{\hat{x}}^{\alpha} s^{-m-1} [h(s) - rK_1] ds - K_1 - K_0 \\ &= \frac{\hat{x}^m}{r} \left(m \int_{\delta_+}^{\hat{x}} s^{-m-1} [h(s) + rK_0] ds + r(K - K_0)\delta_+^{-m} \right). \end{aligned} \quad (99)$$

The second of these expressions and the assumption $K < K_0$ that we have made above imply that $g_1(\hat{x}) < 0$ and (98) holds true if

$$0 \leq h(0) + rK_0.$$

On the other hand, the first expression in (99) implies that $g_1(\hat{x}) \leq 0$ and (98) holds true if

$$h(0) + rK_0 < 0 \quad \text{and} \quad K_0 \geq -K_1 - \frac{m\hat{x}^m}{r} \int_{\hat{x}}^{\alpha} s^{-m-1} [h(s) - rK_1] ds =: K_0^*. \quad (100)$$

A simple inspection of (42)–(43) and (87) that determine δ_+ , α and \hat{x} reveals that these points do not depend on K_0 . Therefore, K_0^* is independent of K_0 . In the context of (100),

$$K < K_0^* < -r^{-1}h(0). \quad (101)$$

The second inequality here follows immediately from the fact that the second identity in (99) implies that $g_1(\hat{x}) < 0$ for all $K_0 \geq -r^{-1}h(0)$. In view of the linear dependence of $g_1(\hat{x})$ on K_0 , we can see that

$$K_0^* > K \quad \Leftrightarrow \quad (\text{if } K_0 = K, \text{ then } g_1(\hat{x}) > 0).$$

Combining this observation with the fact that, if $K = K_0$, then

$$g_1(\hat{x}) \geq g_1(\delta_+) = B\delta_+^n > 0,$$

we obtain the first inequality in (101).

For future reference, we note that the first expression in (99) implies that

$$g_1(\hat{x}) = -\frac{\hat{x}^m}{r} \left(m \int_{\hat{x}}^{\alpha} s^{-m-1} h(s) ds + rK_1\alpha^{-m} + rK_0\hat{x}^{-m} \right).$$

Combining this result with the fact that $g'_1(\hat{x}) = 0$ and (87), we obtain

$$g_1(\hat{x}) = -\frac{\hat{x}^n}{r} \left(n \int_{\hat{x}}^{\alpha} s^{-n-1} h(s) ds + rK_1\alpha^{-n} + rK_0\hat{x}^{-n} \right).$$

Comparing these identities with (36)–(37), we can see that

$$K_0 = K_0^* \Leftrightarrow g_1(\hat{x}) = 0 \Leftrightarrow (\hat{x}, \alpha) \text{ is the solution to (36)–(37)}. \quad (102)$$

□

Proof of Lemma 6. In view of Lemma 4, the system of equations (36)–(37) has a unique solution (α, β) such that $0 < \beta < \alpha$ if and only if $\mathbf{h(0)} < -r\mathbf{K_0}$. To establish conditions under which there exists a unique pair (δ, γ) satisfying the system of equations (46)–(47) and such that $0 < \delta < \gamma < \beta$, we first note that (77) and the assumption that h is increasing imply that

$$h(x) + rK_0 < 0 \quad \text{for all } x \leq \beta. \quad (103)$$

In view of this observation, a simple inspection of (46) reveals that there are no $0 < \delta < \gamma < \beta$ such that $F_1(\delta, \gamma) = 0$ if $K \geq K_0$. Therefore, we assume that $\mathbf{K} < \mathbf{K_0}$ in what follows. Given any $\gamma \in]0, \beta]$, the calculations

$$\begin{aligned} \frac{\partial F_1(\delta, \gamma)}{\partial \delta} &= -m\delta^{-m-1} [h(\delta) + rK_0 + r(K - K_0)] < 0 \quad \text{for all } \delta \in]0, \gamma[, \\ \lim_{\delta \downarrow 0} F_1(\delta, \gamma) &= m \int_0^\gamma s^{-m-1} [h(s) + rK_0] ds > 0 \quad \text{and} \quad F_1(\gamma, \gamma) = r(K - K_0)\gamma^{-m} < 0 \end{aligned}$$

imply that there exists a unique $\delta \in]0, \gamma[$ such that $F_1(\delta, \gamma) = 0$. It follows that there exists a unique mapping $\ell :]0, \beta] \rightarrow]0, \beta[$ such that

$$\ell(\gamma) < \gamma \quad \text{and} \quad F_1(\ell(\gamma), \gamma) = 0 \quad \text{for all } \gamma \in]0, \beta].$$

Differentiating the identity here with respect to γ , we obtain

$$\ell'(\gamma) = \frac{\gamma^{-m-1} [h(\gamma) + rK_0]}{\ell^{-m-1}(\gamma) [h(\ell(\gamma)) + rK]}. \quad (104)$$

In view of these results, we can see that the system of equations (46)–(47) has a unique solution (δ, γ) such that $0 < \delta < \gamma < \beta$ if and only if the equation $F_2(\ell(\gamma), \gamma) = 0$ has a unique solution $\gamma \in]0, \beta[$. To derive conditions under which this is indeed the case, we use (103) and (104) to calculate

$$\frac{dF_2(\ell(\gamma), \gamma)}{d\gamma} = -n[h(\gamma) + rK_0]\gamma^{-m-1} [\ell^{m-n}(\gamma) - \gamma^{m-n}] > 0 \quad \text{for all } \gamma \in]0, \beta[.$$

Combining this result with the identity $\lim_{\gamma \downarrow 0} F_2(\ell(\gamma), \gamma) = -\infty$, which follows from (103) and the assumption that $K < K_0$, we can see that the equation $F_2(\ell(\gamma), \gamma) = 0$ has a unique solution $\gamma \in]0, \beta[$ if and only if

$$F_2(\ell(\beta), \beta) = \int_{\ell(\beta)}^{\infty} s^{-n-1} [h(s) + rK] ds > 0. \quad (105)$$

To derive necessary and sufficient conditions under which this inequality holds true, we fix all other problem data and we parametrise β and $\ell(\beta)$ by $K_0 \in]K, -r^{-1}h(0)[$. Differentiating the identities (36)–(37) with respect to K_0 , we calculate

$$\frac{\partial \beta(K_0)}{\partial K_0} = \frac{\sigma^2(-m\alpha^{n-m} + n\beta^{n-m})\beta}{[h(\beta) + rK_0](\alpha^{n-m} - \beta^{n-m})} < 0,$$

the inequality following thanks to (103). Also, differentiating the identity

$$F_1(\ell(\beta), \beta) \equiv m \int_{\ell(\beta)}^{\beta} s^{-m-1} [h(s) + rK_0] ds + r(K - K_0)\ell^{-m}(\beta) = 0 \quad (106)$$

with respect to K_0 , we can see that

$$\frac{\partial \ell(\beta(K_0); K_0)}{\partial K_0} = \frac{m\beta^{-m-1} [h(\beta(K_0)) + rK_0] \frac{\partial \beta(K_0)}{\partial K_0} - r\beta^{-m}(K_0)}{m\ell^{-m-1}(\beta(K_0); K_0) [h(\ell(\beta(K_0); K_0)) + rK]}.$$

Using these results, we can differentiate the identity (105) with respect to K_0 to obtain

$$\begin{aligned} & \frac{\partial F_2(\ell(\beta(K_0); K_0), \beta(K_0); K_0)}{\partial K_0} \\ &= -\ell^{-(n-m)}(\beta(K_0); K_0) \left(\beta^{-m-1} [h(\beta(K_0)) + rK_0] \frac{\partial \beta(K_0)}{\partial K_0} - \frac{r}{m} \beta^{-m}(K_0) \right) \\ &< 0. \end{aligned} \quad (107)$$

Comparing equation (42) and the second expression in (99) with the expression (105) and the identity (106), and taking into account (102), we can see that

$$F_2(\ell(\beta), \beta) = 0 \quad \Leftrightarrow \quad (\ell(\beta) = \delta_{\dagger} \text{ and } \beta = \hat{x}) \quad \Leftrightarrow \quad K_0 = K_0^*,$$

where δ_{\dagger} , \hat{x} and K_0^* are as in the analysis that established (98) in proof of Lemma 5. In view of this observation and (107), we can see that the inequality in (105) holds true if and only if $K_0 \in]K, K_0^*[$.

To proceed further, we first note that the restriction of w_1 in $[\gamma, \infty[$ as well as the function w_0 are increasing thanks to Lemma 4. We also note that (103) implies that $\Gamma_1 > 0$

and $\Gamma_2 > 0$. The function w_1 is constant in $]0, \delta]$. Furthermore, it is increasing in $[\delta, \gamma]$ because

$$\begin{aligned} w_1'(x) &= m\Gamma_1 x^{m-1} + n\Gamma_2 x^{n-1} + R_h'(x) \\ &= \frac{mx^{m-1}}{\sigma^2(n-m)} \int_{\delta}^x s^{-m-1} [h(s) + rK] ds - \frac{nx^{n-1}}{\sigma^2(n-m)} \int_{\delta}^x s^{-n-1} [h(s) + rK] ds \\ &> 0 \quad \text{for all } x \in]\delta, \gamma], \end{aligned}$$

the inequality following thanks to (103) and the assumption $K < K_0$ that we have already made.

Since the restriction of w_1 in $[\gamma, \infty[$ and the function w_0 satisfy the HJB equation (74)–(75), we will prove that w_1, w_0 are indeed solutions to the HJB equation (6)–(7) if we show that

$$\sigma^2 x^2 w_1''(x) + bxw_1'(x) - rw_1(x) + h(x) \leq 0 \quad \text{for all } x < \delta, \quad (108)$$

$$w_0(x) - w_1(x) - K_0 \leq 0 \quad \text{for all } x \leq \gamma, \quad (109)$$

$$w_1(x) - w_0(x) - K_1 \leq 0 \quad \text{for all } x \leq \gamma, \quad (110)$$

$$-w_1(x) - K \leq 0 \quad \text{for all } x > \delta, \quad (111)$$

$$\text{and} \quad -w_0(x) - K \leq 0 \quad \text{for all } x > 0. \quad (112)$$

The inequality (108) follows immediately from (103), the fact that $\delta < \beta$ and the assumption that $K < K_0$. The inequality (111) follows immediately from the facts that w_1 is increasing and $w_1(\delta) = -K$, while the inequality (112) is equivalent to $\mathbf{K} \geq \mathbf{0}$ because w_0 is also increasing. The inequality (110) for $x \leq \delta$ is equivalent to $Bx^n + K + K_1 \geq 0$, which is true because $B > 0$. For $x < \delta$, (109) holds true if $w_0(\delta) - w_1(\delta) - K_0 \equiv B\delta^n + K - K_0 \leq 0$ because $B > 0$. Therefore, (109) for $x < \delta$ will follow immediately once we establish it for $x \geq \delta$.

The inequalities (109) and (110) for $x \in [\delta, \gamma]$ are equivalent to

$$-K_1 - K_0 \leq g_2(x) \leq 0 \quad \text{for all } x \in [\delta, \gamma], \quad (113)$$

where $g_2(x) = w_0(x) - w_1(x) - K_0$. Using (35), (37) and (48)–(49), we calculate

$$\begin{aligned} g_2(x) &= \frac{x^m}{\sigma^2(n-m)} \int_x^\gamma s^{-m-1} [h(s) + rK_0] ds \\ &\quad - \frac{x^n}{\sigma^2(n-m)} \int_x^\gamma s^{-n-1} [h(s) + rK_0] ds \end{aligned} \quad (114)$$

$$\begin{aligned} \text{and } g_2'(x) &= \frac{mx^{m-1}}{\sigma^2(n-m)} \int_x^\gamma s^{-m-1} [h(s) + rK_0] ds \\ &\quad - \frac{nx^{n-1}}{\sigma^2(n-m)} \int_x^\gamma s^{-n-1} [h(s) + rK_0] ds > 0 \quad \text{for all } x \in [\delta, \gamma[, \end{aligned} \quad (115)$$

the inequality following thanks to (103). Combining the fact that $g_2(x)$ is strictly increasing as x increases from δ to γ with the inequality $g_2(\delta) \geq -K_1 - K_0$, which follows from (110) for $x \leq \delta$ that we have established above, and the identity $g_2(\gamma) = 0$, we obtain (113). \square

Proof of Lemma 7. As we have seen at the beginning of the proof of Lemma 5, equation (42) has a unique solution $\delta_{\dagger} > 0$ if and only if $\mathbf{h}(\mathbf{0}) + \mathbf{rK} < \mathbf{0}$, in which case,

$$h(x) + rK < 0 \quad \text{for all } x \leq \delta_{\dagger}, \quad (116)$$

and $A > 0$. We also note that the inequality $\sigma^2 x^2 w_0''(x) + b x w_0'(x) - r w_0(x) \leq 0$, which is associated with the HJB equation (7), can be true for $x < \zeta$ if and only if $K \leq 0$. If $K = 0$, then (42) and (52)–(53) imply that α and δ_{\dagger} should satisfy

$$\int_{\alpha}^{\infty} s^{-n-1} [h(s) - rK_1] ds = 0 \quad \text{and} \quad \int_{\delta_{\dagger}}^{\alpha} s^{-m-1} [h(s) - rK_1] ds = -\frac{rK_1}{m} \delta_{\dagger}^{-m} > 0,$$

which is not possible because h is increasing. We therefore assume that $\mathbf{K} < \mathbf{0}$ in what follows. In particular, this assumption implies that $\Delta_1 > 0$ and $\Delta_2 > 0$.

To establish the solvability of (52)–(53), we note that the calculations

$$\lim_{\zeta \downarrow 0} G_2(\delta_{\dagger}, \zeta, \alpha) = -\infty \quad \text{and} \quad \frac{\partial G_2(\delta_{\dagger}, \zeta, \alpha)}{\partial \zeta} = -nrK\zeta^{-n-1} > 0$$

ensure that, given any $\alpha > \delta_{\dagger}$ fixed, the equation $G_2(\delta_{\dagger}, \zeta, \alpha) = 0$ has a unique solution $\zeta \in]0, \alpha[$ if and only if

$$H_1(\alpha) := G_2(\delta_{\dagger}, \alpha, \alpha) = -rK_1\alpha^{-n} - n \int_{\delta_{\dagger}}^{\alpha} s^{-n-1} [h(s) + rK] ds > 0.$$

In view of the calculations

$$\begin{aligned} H_1(\delta_{\dagger}) &= -rK_1\delta_{\dagger}^{-n} < 0, \quad \lim_{\alpha \rightarrow \infty} H_1(\alpha) = 0 \\ \text{and} \quad H_1'(\alpha) &= -n\alpha^{-n-1} [h(\alpha) + rK - rK_1] \begin{cases} > 0, & \text{if } \alpha \in]\delta_{\dagger}, \alpha[\\ < 0, & \text{if } \alpha \in]\tilde{\alpha}, \infty[\end{cases}, \end{aligned}$$

where

$$\begin{aligned} \delta_{\dagger} < \alpha &:= \inf \{x > 0 \mid h(x) + rK - rK_1 \geq 0\} \\ &\leq \inf \{x > 0 \mid h(x) + rK - rK_1 > 0\} =: \tilde{\alpha}, \end{aligned} \quad (117)$$

we can see that there exists a unique $\hat{\alpha} \in]\delta_{\dagger}, \alpha[$ such that

$$H_1(\alpha) \begin{cases} < 0, & \text{if } \alpha \in [\delta_{\dagger}, \hat{\alpha}[\\ > 0, & \text{if } \alpha \in]\hat{\alpha}, \infty[\end{cases}.$$

It follows that there exists a function $\ell :]\hat{\alpha}, \infty[\rightarrow]0, \infty[$ such that

$$G_2(\delta_{\dagger}, \ell(\alpha), \alpha) = 0 \text{ and } \ell(\alpha) < \alpha \text{ for all } \alpha \in]\hat{\alpha}, \infty[, \quad \text{and} \quad \lim_{\alpha \downarrow \hat{\alpha}} \ell(\alpha) = \hat{\alpha}. \quad (118)$$

Furthermore,

$$\ell(\alpha) < \delta_{\dagger} \Leftrightarrow G_2(\delta_{\dagger}, \delta_{\dagger}, \alpha) > 0 \quad \text{and} \quad \ell(\alpha) = \delta_{\dagger} \Leftrightarrow G_2(\delta_{\dagger}, \delta_{\dagger}, \alpha) = 0. \quad (119)$$

Differentiating the identity $G_2(\delta_{\dagger}, \ell(\alpha), \alpha) = 0$ with respect to α , we obtain

$$\ell'(\alpha) = -\frac{1}{rK} \ell^{m+1}(\alpha) \alpha^{-n-1} [h(\alpha) - rK_1]. \quad (120)$$

In view of the analysis thus far, we will show that there exist unique $0 < \zeta < \alpha$ such that (52) and (53) hold true if we prove that there exists a unique $\alpha > \hat{\alpha}$ such that $G_1(\delta_{\dagger}, \ell(\alpha), \alpha) = 0$. To this end, we note that

$$\lim_{\alpha \downarrow \hat{\alpha}} G_1(\delta_{\dagger}, \ell(\alpha), \alpha) = G_1(\delta_{\dagger}, \hat{\alpha}, \hat{\alpha}) = rK_1 \delta_{\dagger}^{-m} + m \int_{\delta_{\dagger}}^{\hat{\alpha}} s^{-m-1} [h(s) + rK - rK_1] ds > 0,$$

the inequality following because $\hat{\alpha} \in]\delta_{\dagger}, \alpha[$, where α is defined by (117). In view of the inequality $-rK \ell^{-m}(\alpha) < -rK \alpha^{-m}$ and (68) in Lemma 10, we can see that

$$\lim_{\alpha \rightarrow \infty} G_1(\delta_{\dagger}, \ell(\alpha), \alpha) \leq \lim_{\alpha \rightarrow \infty} \left(rK_1 \delta_{\dagger}^{-m} + m \int_{\delta_{\dagger}}^{\alpha} s^{-m-1} [h(s) + rK - rK_1] ds \right) = -\infty.$$

Combining these results with the observation that

$$\begin{aligned} \frac{\partial G_1(\delta_{\dagger}, \ell(\alpha), \alpha)}{\partial \alpha} &= m \alpha^{-m-1} [h(\alpha) - rK_1] + m r K \ell^{-m-1}(\alpha) \ell'(\alpha) \\ &= m \alpha^{-n-1} [h(\alpha) - rK_1] [\alpha^{n-m} - \ell^{n-m}(\alpha)] \\ &\quad \begin{cases} > 0, & \text{if } \hat{\alpha} < \alpha \text{ and } \alpha \in]\hat{\alpha}, \alpha[\\ < 0, & \text{if } \alpha \in]\hat{\alpha} \vee \bar{\alpha}, \infty[\end{cases} \end{aligned}$$

where we have used (120) and the definitions

$$\alpha := \inf \{x > 0 \mid h(x) - rK_1 \geq 0\} \leq \inf \{x > 0 \mid h(x) - rK_1 > 0\} =: \bar{\alpha} < \tilde{\alpha},$$

we can see that equation $G_1(\delta_{\dagger}, \ell(\alpha), \alpha) = 0$ has a unique solution $\alpha > \hat{\alpha}$ such that

$$h(x) - rK_1 > 0 \quad \text{for all } x \geq \alpha. \quad (121)$$

We now investigate under what conditions $\zeta \leq \delta_{\dagger}$ or $\zeta > \delta_{\dagger}$. To this end, we calculate

$$G_2(\delta_{\dagger}, \delta_{\dagger}, \alpha) = -n \int_{\delta_{\dagger}}^{\alpha} s^{-n-1} h(s) ds - rK_1 \alpha^{-n}. \quad (122)$$

In view of this expression and (119), we can see that

$$h(\delta_{\dagger}) \geq 0 \quad \Rightarrow \quad \delta_{\dagger} < \zeta.$$

If $h(\delta_{\dagger}) < 0$, then we fix all other problem data and we parametrise G_1 , G_2 , ζ and α by $K_1 > 0$ (note that δ_{\dagger} does not depend on K_1). Differentiating (52) and (53), and eliminating $\frac{\partial \zeta(K_1)}{\partial K_1}$, we calculate

$$\frac{\partial \alpha(K_1)}{\partial K_1} = \frac{\sigma^2 \alpha(K_1) [n \zeta^{m-n}(K_1) - m \alpha^{m-n}(K_1)]}{[\zeta^{m-n}(K_1) - \alpha^{m-n}(K_1)] [h(\alpha(K_1)) - r K_1]} > 0, \quad (123)$$

the inequality following thanks to (121) and the fact that $\zeta < \alpha$. Furthermore, (121) implies that

$$\lim_{K_1 \rightarrow \infty} \alpha(K_1) = \infty. \quad (124)$$

Using (123), we calculate

$$\frac{\partial G_2(\delta_{\dagger}, \delta_{\dagger}, \alpha(K_1); K_1)}{\partial K_1} = -n \alpha^{-n-1}(K_1) [h(\alpha(K_1)) - r K_1] \frac{\partial \alpha(K_1)}{\partial K_1} - r \alpha(K_1) < 0.$$

In view of (124), we can see that

$$\lim_{K_1 \rightarrow \infty} G_2(\delta_{\dagger}, \delta_{\dagger}, \alpha(K_1); K_1) \leq -n \int_{\delta_{\dagger}}^{\infty} s^{-n-1} h(s) ds = r K \delta_{\dagger}^{-n} < 0,$$

where we have also used (42) and the assumption $K < 0$ that we have made above. In light of (119), it follows that, if

$$\lim_{K_1 \downarrow 0} G_2(\delta_{\dagger}, \delta_{\dagger}, \alpha(K_1); K_1) > 0, \quad (125)$$

then

$$h(\delta_{\dagger}) < 0$$

$$\Rightarrow \text{there exists a unique } K_1^{\dagger} > 0 \text{ such that } \left\{ \begin{array}{ll} \zeta < \delta_{\dagger} & \text{for all } K_1 \in]0, K_1^{\dagger}[\\ \delta_{\dagger} < \zeta & \text{for all } K_1 \in]K_1^{\dagger}, \infty[\end{array} \right\}. \quad (126)$$

This analysis also establishes (56)–(58) if (125) holds true whenever $h(\delta_{\dagger}) < 0$. For future reference, we note that

$$\text{if } h(\delta_{\dagger}) < 0, \text{ then } G_2(\delta_{\dagger}, \delta_{\dagger}, \alpha(K_1^{\dagger}); K_1^{\dagger}) = 0. \quad (127)$$

Furthermore, K_1^{\dagger} does not depend on K_1 itself or K_0 .

To prove that (125) is indeed true, we first note that the analysis of the solvability of (52)–(53) remains true for $K_1 = 0$. In particular, if we define $\zeta_0 = \lim_{K_1 \downarrow 0} \zeta(K_1)$ and $\alpha_0 = \lim_{K_1 \downarrow 0} \alpha(K_1)$, then

$$\delta_{\dagger} < \alpha_0, \quad \zeta_0 < \alpha_0, \quad \bar{\alpha}_0 := \inf\{x > 0 \mid h(x) > 0\} < \alpha_0,$$

$$\begin{aligned} L_1(\zeta_0, \alpha_0) &:= \lim_{K_1 \downarrow 0} G_1(\delta_{\dagger}, \zeta(K_1), \alpha(K_1); K_1) \\ &\equiv m \int_{\delta_{\dagger}}^{\alpha_0} s^{-m-1} h(s) ds + rK\delta_{\dagger}^{-m} - rK\zeta_0^{-m} = 0 \\ \text{and } L_2(\zeta_0, \alpha_0) &:= \lim_{K_1 \downarrow 0} G_2(\delta_{\dagger}, \zeta(K_1), \alpha(K_1); K_1) \\ &\equiv -n \int_{\delta_{\dagger}}^{\alpha_0} s^{-n-1} h(s) ds - rK\delta_{\dagger}^{-n} + rK\zeta_0^{-n} = 0. \end{aligned}$$

In view of Lemma 14, the identities here cannot be satisfied for $\zeta_0 = \delta_{\dagger}$. To prove that $\zeta_0 < \delta_{\dagger}$, which is equivalent to (125), we argue by contradiction and we assume that $\delta_{\dagger} < \zeta_0$. The calculations

$$\frac{\partial L_1(\zeta, \alpha)}{\partial \alpha} = m\alpha^{-m-1}h(\alpha) \begin{cases} \geq 0, & \text{if } \alpha \leq \bar{\alpha}_0 \\ < 0, & \text{if } \alpha > \bar{\alpha}_0 \end{cases} \quad \text{and} \quad \lim_{\alpha \rightarrow \infty} L_1(\zeta, \alpha) = -\infty,$$

where the last one follows from (68) in Lemma 10, imply that, given any $\zeta > \delta_{\dagger}$, there exists a unique $\alpha > \zeta$ such that $L_1(\zeta, \alpha) = 0$ if and only if

$$L_1(\zeta, \zeta) = m \int_{\delta_{\dagger}}^{\zeta} s^{-m-1} [h(s) + rK] ds > 0. \quad (128)$$

In view of our assumptions on h , (116) and the fact that $\lim_{\zeta \rightarrow \infty} L_1(\zeta, \zeta) = -\infty$, which follows from (68) in Lemma 10, we can see that there exists a unique

$$\zeta^{\dagger} > \inf\{x > 0 \mid h(x) + rK > 0\} > \delta_{\dagger}$$

such that (128) holds true if and only if $\zeta \in]\delta_{\dagger}, \zeta^{\dagger}[$. It follows that there exists a function $\lambda :]\delta_{\dagger}, \zeta^{\dagger}[\rightarrow]\bar{\alpha}_0, \infty[$ such that

$$L_1(\zeta, \lambda(\zeta)) = 0 \text{ and } \zeta < \lambda(\zeta) \text{ for all } \zeta \in]\delta_{\dagger}, \zeta^{\dagger}[, \quad \text{and} \quad \lambda(\zeta_0) = \alpha_0. \quad (129)$$

Differentiating the identity $L_1(\zeta, \lambda(\zeta)) = 0$ with respect to ζ , we calculate

$$\lambda'(\zeta) = -\frac{rK\zeta^{-m-1}}{\lambda^{-m-1}(\zeta)h(\lambda(\zeta))}.$$

Using this result, we obtain

$$\frac{dL_2(\zeta, \lambda(\zeta))}{d\zeta} = -nrK\zeta^{-m-1}[\zeta^{-(n-m)} - \lambda^{-(n-m)}(\zeta)] > 0.$$

Combining this calculation with (129) and the limit

$$\lim_{\zeta \downarrow \delta_{\dagger}} L_2(\zeta, \lambda(\zeta)) = -n \int_{\delta_{\dagger}}^{\lambda(\delta_{\dagger})} s^{-n-1} h(s) ds \geq 0,$$

where the inequality follows from the fact that $L_1(\delta_{\dagger}, \lambda(\delta_{\dagger})) = 0$ and Lemma 14, we can see that there exist no $\zeta_0 < \alpha_0$ such that $\zeta_0 > \delta_{\dagger}$ and $L_1(\zeta_0, \alpha_0) = L_2(\zeta_0, \alpha_0) = 0$, which establishes the required contradiction.

To show that w_1 , w_0 are increasing, it suffices to prove that w_1 is increasing in $[\delta_{\dagger}, \infty[$ and w_0 is increasing in $[\zeta, \alpha]$. The first of these claims follows from the calculation

$$w_1'(x) = R_h'(x) + mA x^{m-1} = x^{m-1} [x^{-m+1} R_h'(x) - \delta_{\dagger}^{-m+1} R_h'(\delta_{\dagger})] > 0 \quad \text{for all } x > \delta_{\dagger},$$

where we have used the expression for A given by (40), the identity (42), and Lemma 13. To establish the second claim, we use (54) and (55) to calculate

$$w_0'(x) = -\frac{rK}{\sigma^2(n-m)x} \left[\left(\frac{x}{\zeta} \right)^n - \left(\frac{x}{\zeta} \right)^m \right] > 0 \quad \text{for all } x \in]\zeta, \alpha[.$$

To show that the functions w_1 , w_0 defined by (50), (51) satisfy the inequalities associated with the HJB equation (6)–(7), we need to show that

$$\sigma^2 x^2 w_1''(x) + b x w_1'(x) - r w_1(x) + h(x) \leq 0 \quad \text{for all } x < \delta_{\dagger}, \quad (130)$$

$$\sigma^2 x^2 w_0''(x) + b x w_0'(x) - r w_0(x) \leq 0 \quad \text{for all } x \in]0, \zeta[\cup]\alpha, \infty[, \quad (131)$$

$$w_0(x) - w_1(x) - K_0 \leq 0 \quad \text{for all } x > 0, \quad (132)$$

$$w_1(x) - w_0(x) - K_1 \leq 0 \quad \text{for all } x \leq \alpha, \quad (133)$$

$$-w_1(x) - K \leq 0 \quad \text{for all } x > \delta_{\dagger} \quad (134)$$

$$\text{and } -w_0(x) - K \leq 0 \quad \text{for all } x > \zeta. \quad (135)$$

The inequality (130) is equivalent to $h(x) \leq -rK$ for all $x \leq \delta_{\dagger}$, which is true thanks to (116). The inequality (131) is trivial for $x < \zeta$ and follows immediately from (121) for $x > \alpha$. The inequalities (132) and (133) for $x \leq \delta_{\dagger} \wedge \zeta$ are equivalent to $K_0 \geq 0$ and $K_1 \geq 0$, respectively, which are true by assumption, while (132) for $x \geq \alpha$ is also implied by the assumption that $K_1, K_0 > 0$. The inequalities (134) and (135) follow from the fact that w_1 , w_0 are increasing and the identities $w_1(\delta_{\dagger}) = w_0(\zeta) = -K$. If $\delta_{\dagger} < \zeta$, then (132) for $x \in [\delta_{\dagger}, \zeta]$ is equivalent to $w_1(x) \geq -K - K_0$, which is true, while (133) for $x \in [\delta_{\dagger}, \zeta]$ will follow as soon as we establish it for $x \in [\zeta, \alpha]$ below because w_1 is increasing.

The inequalities (132) and (133) for $x \in [\zeta, \alpha]$ are equivalent to

$$-K_1 - K_0 \leq g_1(x) \leq 0 \quad \text{for all } x \in [\zeta, \alpha], \quad (136)$$

where $g_1(x) = w_0(x) - w_1(x) - K_0$. Using (40) and (54)–(55), we can see that g_1 and g'_1 admit the expressions given by (86) and (87). Furthermore,

$$g_1(\alpha) = -K_1 - K_0 \quad \text{and} \quad g'_1(\alpha) = 0. \quad (137)$$

Proof of (136) when $\delta_{\dagger} \leq \zeta$ (i.e., when $\mathbf{h}(\delta_{\dagger}) \geq \mathbf{0}$ or $\mathbf{h}(\delta_{\dagger}) < \mathbf{0}$ and $\mathbf{K}_1 \geq \mathbf{K}_1^{\dagger}$). In view of the expression for A given by (40), the identity (42), and Lemma 13, we can see that

$$\begin{aligned} g'_1(\zeta) &= -R'_h(\zeta) - mA\zeta^{m-1} \\ &= -\zeta^{m-1}[\zeta^{-m+1}R'_h(\zeta) - \delta_{\dagger}^{-m+1}R'_h(\delta)] \begin{cases} = 0, & \text{if } \zeta = \delta_{\dagger} \\ < 0, & \text{if } \delta_{\dagger} < \zeta \end{cases}, \end{aligned} \quad (138)$$

Also, since w_1 is increasing,

$$g_1(\zeta) \leq -K - w_1(\delta_{\dagger}) - K_0 = -K_0 < 0.$$

Combining these inequalities with (137) and Lemma 11, we can see that $g_1(x)$ is decreasing from $g_1(\zeta) < 0$ to $-K_1 - K_0$ as x increases from ζ to α , and (136) follows.

Proof of (136) when $\zeta < \delta_{\dagger}$ (i.e., when $\mathbf{h}(\delta_{\dagger}) < \mathbf{0}$ and $\mathbf{K}_1 < \mathbf{K}_1^{\dagger}$). Since w_0 is strictly increasing in $] \zeta, \alpha[$,

$$g_1(\zeta) = -K_0 > -K_1 - K_0 \quad \text{and} \quad g'_1(\delta_{\dagger}) = w'_0(\delta_{\dagger}) > 0.$$

Combining these inequalities with (137) and Lemma 11, we can see that there exists a unique

$$\hat{x} \in]\delta_{\dagger}, \alpha[, \quad (139)$$

such that

$$g'_1(x) \begin{cases} > 0 & \text{for all } x \in [\zeta, \hat{x}] \\ < 0 & \text{for all } x \in]\hat{x}, \alpha[\end{cases}.$$

In particular, g_1 has a unique maximum in $[\zeta, \alpha]$ and (136) holds true if and only if $g_1(\hat{x}) \leq 0$. Using the expressions (86), (87) of g_1 , g'_1 and the identity $\sigma^2 mn = -r$, we calculate

$$\begin{aligned} g_1(\hat{x}) &= -\frac{n\hat{x}^n}{r} \int_{\hat{x}}^{\alpha} s^{-n-1} [h(s) - rK_1] ds - K_1 - K_0 \\ &= -\frac{m\hat{x}^m}{r} \int_{\hat{x}}^{\alpha} s^{-m-1} [h(s) - rK_1] ds - K_1 - K_0. \end{aligned} \quad (140)$$

The second of these expressions and (52) imply that

$$\begin{aligned} g_1(\hat{x}) &= \frac{\hat{x}^m}{r} \left(m \int_{\delta_{\dagger}}^{\hat{x}} s^{-m-1} [h(s) + rK_0] ds + r(K - K_0)\delta_{\dagger}^{-m} - rK\zeta^{-m} \right) \\ &< \frac{\hat{x}^m}{r} \left(m \int_{\delta_{\dagger}}^{\hat{x}} s^{-m-1} [h(s) + rK_0] ds - rK_0\delta_{\dagger}^{-m} \right). \end{aligned}$$

This inequality reveals that $g_1(\hat{x}) < 0$ and (136) holds true if

$$0 \leq h(0) + rK_0.$$

On the other hand, the first expression in (140) implies that $g_1(\hat{x}) \leq 0$ and (136) holds true if

$$h(0) + rK_0 < 0 \quad \text{and} \quad K_0 \geq -K_1 - \frac{n\hat{x}^n}{r} \int_{\hat{x}}^{\alpha} s^{-n-1} [h(s) - rK_1] ds =: K_0^{\dagger} > 0. \quad (141)$$

A simple inspection of (42), (52)–(53) and (87) that determine δ_{\dagger} , α and \hat{x} reveals that these points do not depend on K_0 . Therefore, K_0^{\dagger} is independent of K_0 . To see the last inequality in (141), we note that $g_1(\cdot) + K_0$ does not depend on K_0 and has a unique global maximum in $[\delta_{\dagger}, \alpha]$ at $\hat{x} \in]\delta_{\dagger}, \alpha[$. Therefore,

$$K_0^{\dagger} = g_1(\hat{x}) + K_0 > g_1(\delta_{\dagger}) + K_0 = \frac{\delta_{\dagger}^n}{r} G_2(\delta_{\dagger}, \delta_{\dagger}, \alpha) > 0, \quad (142)$$

the last inequality following from (119) and the fact that $\zeta = \ell(\alpha) < \delta_{\dagger}$ in this part of the analysis.

For future reference, we note that (140) implies that,

$$\begin{aligned} g_1(\hat{x}) &= -\frac{\hat{x}^n}{r} \left(n \int_{\hat{x}}^{\alpha} s^{-n-1} h(s) ds + rK_1\alpha^{-n} + rK_0\hat{x}^{-n} \right) \\ &= -\frac{\hat{x}^m}{r} \left(m \int_{\hat{x}}^{\alpha} s^{-m-1} h(s) ds + rK_1\alpha^{-m} + rK_0\hat{x}^{-m} \right). \end{aligned}$$

Comparing these identities with (36)–(37), we can see that, if $h(\delta_{\dagger}) < 0$, $K_1 \in]0, K_1^{\dagger}]$ and $h(0) + rK_0 < 0$, then

$$K_0 = K_0^{\dagger} \Leftrightarrow g_1(\hat{x}) = 0 \Leftrightarrow (\hat{x}, \alpha) \text{ is the solution to (36)–(37)}. \quad (143)$$

Furthermore, if we fix all other problem data and we parametrise \hat{x} , ζ and K_0^{\dagger} by K_1 , then (127), (138) and a calculation similar to the one in (142) imply that

$$\hat{x}(K_1^{\dagger}) = \zeta(K_1^{\dagger}) = \delta_{\dagger} \text{ and } \lim_{K_1 \uparrow K_1^{\dagger}} K_0^{\dagger}(K_1) = 0, \quad (144)$$

(42), (53) and (143) imply that

$$n \int_{\hat{x}(K_1)}^{\infty} s^{-n-1} [h(s) + rK_0^\dagger(K_1)] ds + rK\zeta^{-n}(K_1) = 0, \quad (145)$$

while (52) and (143) imply that

$$\begin{aligned} m \int_0^{\hat{x}(K_1)} s^{-m-1} [h(s) + rK_0^\dagger(K_1)] ds \\ - m \int_0^{\delta_\dagger} s^{-m-1} [h(s) + rK] ds - rK\zeta^{-m}(K_1) = 0. \end{aligned} \quad (146)$$

□

Proof of Lemma 8. In view of Lemma 4, the system of equations (36)–(37) has a unique solution (α, β) such that $0 < \beta < \alpha$ if and only if $\mathbf{h}(\mathbf{0}) + r\mathbf{K}_0 < \mathbf{0}$, in which case,

$$h(x) + rK_0 < 0 \text{ for all } x \leq \beta \quad \text{and} \quad h(x) - rK_1 > 0 \text{ for all } x \geq \alpha. \quad (147)$$

Equations (36)–(37) imply that (62) is equivalent to

$$n \int_{\alpha}^{\infty} s^{-n-1} [h(s) - rK_1] ds + rK\zeta^{-n} = 0. \quad (148)$$

In view of the second of the inequalities in (147), we can see that there is no $\zeta > 0$ such that (148) holds true unless $\mathbf{K} < \mathbf{0}$. On the other hand, if $K < 0$, then it is straightforward to see that there exists a unique $\zeta \in]0, \alpha[$ such that (148) holds true.

For future reference, we note that the first of the inequalities in (147) and the assumption $K < 0$ imply that $h(0) + rK < 0$. Therefore, there exists a unique $\delta_\dagger > 0$ such that

$$\int_x^{\infty} s^{-n-1} [h(s) + rK] ds \begin{cases} < 0, & \text{for all } x \in]0, \delta_\dagger[\\ > 0, & \text{for all } x \in]\delta_\dagger, \infty[\end{cases}. \quad (149)$$

To establish the required solvability of (61) and (63), we first fix any $\gamma \in]0, \beta[$ and we look for $\delta \in]0, \gamma[$ such that $G_3(\delta, \gamma, \beta) = 0$. Combining the limit

$$\lim_{\delta \downarrow 0} G_3(\delta, \gamma, \beta) = -\infty,$$

which follows from (69) in Lemma 10, with the calculation

$$\frac{\partial G_3(\delta, \gamma, \beta)}{\partial \delta} = -n\delta^{-n-1} [h(\delta) + rK_0 - r(K_0 - K)] > 0 \quad \text{for all } \delta \in]0, \gamma[,$$

where the inequality follows thanks to (147) and the assumption that $K < 0 < K_0$, we can see that there exists $\delta \in]0, \gamma[$ such that $G_3(\delta, \gamma, \beta) = 0$ if and only if

$$G_3(\gamma, \gamma, \beta) = n \int_{\beta}^{\infty} s^{-n-1} [h(s) + rK] ds - r(K_0 - K)\gamma^{-n} + r(K_0 - K)\beta^{-n} > 0. \quad (150)$$

In view of the calculations

$$\lim_{\gamma \downarrow 0} G_3(\gamma, \gamma, \beta) = -\infty \quad \text{and} \quad \frac{\partial G_3(\gamma, \gamma, \beta)}{\partial \gamma} = nr(K_0 - K)\gamma^{-n-1} > 0,$$

there exists a unique $\hat{\gamma} \in]0, \beta[$ such that (150) holds true for all $\gamma \in]\hat{\gamma}, \beta[$ if and only if

$$H_2(\beta) := G_3(\beta, \beta, \beta) = n \int_{\beta}^{\infty} s^{-n-1} [h(s) + rK] ds > 0 \quad \Leftrightarrow \quad \delta_{\dagger} < \beta, \quad (151)$$

where δ_{\dagger} is as in (149). It follows that, if the problem data is such that (151) holds true, then $G_3(\delta, \gamma, \beta) = 0$ defines uniquely a mapping $\ell :]\hat{\gamma}, \beta[\rightarrow]0, \beta[$, such that

$$\begin{aligned} G_3(\ell(\gamma), \gamma, \beta) &= 0 \quad \text{and} \quad \ell(\gamma) < \gamma \quad \text{for all } \gamma \in]\hat{\gamma}, \beta[, \\ \ell(\hat{\gamma}) &:= \lim_{\gamma \downarrow \hat{\gamma}} \ell(\gamma) = \hat{\gamma} \quad \text{and} \quad \ell(\beta) = \delta_{\dagger}. \end{aligned}$$

Differentiating the identity $G_3(\ell(\gamma), \gamma, \beta) = 0$ with respect to γ , we derive the expression

$$\ell'(\gamma) = \frac{\gamma^{-n-1} [h(\gamma) + rK_0]}{\ell^{-n-1}(\gamma) [h(\ell(\gamma)) + rK]} > 0. \quad (152)$$

Furthermore, comparing the identity

$$\lim_{\gamma \downarrow \hat{\gamma}} G_3(\gamma, \gamma, \beta) = n \int_{\beta}^{\infty} s^{-n-1} [h(s) + rK_0] ds - r(K_0 - K)\hat{\gamma}^{-n} = 0$$

with equation (62) that $\zeta > 0$ satisfies, we can see that $K(\zeta^{-n} - \hat{\gamma}^{-n}) = -K_0\hat{\gamma}^{-n}$. It follows that $\zeta < \hat{\gamma}$ because $K < 0 < K_0$. We conclude this part of the analysis with the observation that

$$\zeta < \ell(\gamma) < \delta_{\dagger} \quad \text{for all } \gamma \in]\hat{\gamma}, \beta[, \quad (153)$$

where we have taken into account that $\zeta < \hat{\gamma} = \ell(\hat{\gamma})$ and the fact that ℓ is strictly increasing.

To determine conditions under which there exists a unique $\gamma \in]\hat{\gamma}, \beta[$ such that $G_5(\zeta, \ell(\gamma), \gamma) = 0$ if (151) holds true, we first note that

$$G_5(\zeta, \ell(\hat{\gamma}), \hat{\gamma}) = G_5(\zeta, \hat{\gamma}, \hat{\gamma}) = -rK_0\hat{\gamma}^{-m} + rK(\hat{\gamma}^{-m} - \zeta^{-m}) < 0,$$

the inequality following thanks to (153) and the fact that $K < 0 < K_0$. Combining this observation with the calculation

$$\frac{\partial G_5(\zeta, \ell(\gamma), \gamma)}{\partial \gamma} = m\gamma^{-n-1}[h(\gamma) + rK_0](\gamma^{n-m} - \ell^{n-m}(\gamma)) > 0,$$

where we have used (152), we can see that there exists a unique $\gamma \in]\hat{\gamma}, \beta[$ such that $G_5(\zeta, \ell(\gamma), \gamma) = 0$ if and only if

$$G_5(\zeta, \ell(\beta), \beta) = G_5(\zeta, \delta_{\dagger}, \beta) > 0. \quad (154)$$

To derive conditions under which (151) and (154) hold true, we first note that (147) implies that $h(\beta) < 0$. Therefore, (151) can be true only if $\mathbf{h}(\delta_{\dagger}) < \mathbf{0}$. If we fix all other problem data and we parametrise α , β , ζ and δ by K_1 and K_0 , then (143) and (144) imply that

$$\lim_{K_0 \downarrow 0} H_2(\beta(K_1^{\dagger}, K_0)) = H_2(\hat{x}(K_1^{\dagger})) = n \int_{\delta_{\dagger}}^{\infty} s^{-n-1} [h(s) + rK] ds = 0. \quad (155)$$

Furthermore, differentiating (36)–(37) and using the resulting expressions, we obtain

$$\frac{\partial H_2(\beta(K_1, K_0))}{\partial K_1} = -\frac{\sigma^2(n-m)[h(\beta) + rK]\alpha^{-m}}{[h(\beta) + rK_0](\alpha^{n-m} - \beta^{n-m})} < 0 \quad (156)$$

and

$$\frac{\partial H_2(\beta(K_1, K_0))}{\partial K_0} = -\frac{\sigma^2[h(\beta) + rK](-m\alpha^{n-m} + n\beta^{n-m})\beta^{-n}}{[h(\beta) + rK_0](\alpha^{n-m} - \beta^{n-m})} < 0. \quad (157)$$

These calculations imply that $H_2(\beta(K_1, K_0)) < 0$ for all $K_1 \geq K_1^{\dagger}$ and $K_0 > 0$. On the other hand, (139) and (143) imply that $\beta(K_1, K_0^{\dagger}) = \hat{x}(K_1) > \delta_{\dagger}$ for all $K_1 < K_1^{\dagger}$, which, combined with the equivalences in (143), (151) and the inequality (157), implies that $H_2(\beta(K_1, K_0)) > 0$ for all $K_1 < K_1^{\dagger}$ and $K_0 \in]0, K_0^{\dagger}[$.

In view of the results derived above, we will conclude this part of the analysis if we show that, given any $\mathbf{K}_1 < \mathbf{K}_1^{\dagger}$, (154) holds true if and only if $\mathbf{K}_0 \in]\mathbf{0}, \mathbf{K}_0^{\dagger}[$. In the context of the conditions $h(\delta_{\dagger}) < 0$ and $K_1 < K_1^{\dagger}$, we can see that a straightforward comparison of (62), which defines ζ (see the analysis in the first paragraph of this proof), and (145) reveals that the free-boundary point $\zeta = \zeta(K_1, K_0)$ in this lemma identifies with the free-boundary point $\zeta = \zeta(K_1)$ in Lemma 7 if $K_0 = K_0^{\dagger}$. This observation and a comparison of (63), (146) reveal that

$$G_5(\zeta(K_1, K_0^{\dagger}), \delta_{\dagger}, \beta(K_1, K_0^{\dagger}); K_0^{\dagger}) = 0.$$

Combining this result with the calculation

$$\frac{\partial G_5(\zeta(K_1, K_0), \delta_+, \beta(K_1, K_0); K_0)}{\partial K_0} = \sigma^2 m(n-m) \beta^{-m} \frac{\alpha^{n-m} - \zeta^{n-m}}{\alpha^{n-m} - \beta^{n-m}} < 0,$$

we can see that (154) holds true if and only if $K_0 \in]0, K_0^\dagger[$.

To show that w_1, w_0 are increasing, it suffices to prove that w_0 is increasing in $[\zeta, \alpha]$ and w_1 is increasing in $[\delta, \gamma] \cup [\beta, \infty[$. The first of these claims follows immediately from the calculation

$$w'_0(x) = -\frac{rK}{\sigma^2(n-m)x} \left[\left(\frac{x}{\zeta} \right)^n - \left(\frac{x}{\zeta} \right)^m \right] > 0 \quad \text{for all } x \in]\zeta, \alpha],$$

where we have used (66) and the assumption $K < 0$ that we have made above in this proof. Using (64)–(65), we calculate

$$w'_1(x) = \frac{mx^{m-1}}{\sigma^2(n-m)} \int_\delta^x s^{-m-1} [h(s) + rK] ds - \frac{nx^{n-1}}{\sigma^2(n-m)} \int_\delta^x s^{-n-1} [h(s) + rK] ds,$$

for $x \in [\delta, \gamma]$. This expression, the fact that $w'_1(\gamma) = w'_0(\gamma) > 0$ and Lemma 12 for $\nu = \delta$, $L = rK$ and $q = w'_1$ (see also (149), (153) and recall that $\delta = \ell(\gamma)$) imply that $w'_1(x) > 0$ for all $x \in]\delta, \gamma]$. To prove that w_1 is increasing in $[\beta, \infty[$, we first note that the inequality $w'_1(\beta) = w'_0(\beta) > 0$ implies that $mA > -\beta^{-m+1}R'_h(\beta)$. In view of this observation, we can see that

$$w'_1(x) = R'_h(x) + mA x^{m-1} > x^{m-1} [x^{-m+1}R'_h(x) - \beta^{-m+1}R'_h(\beta)] > 0 \quad \text{for all } x > \beta,$$

the second inequality following by Lemma 13.

To show that w_1 and w_0 satisfy the HJB equation (6)–(7), we need to prove that

$$\sigma^2 x^2 w''_1(x) + bxw'_1(x) - rw_1(x) + h(x) \leq 0 \quad \text{for all } x \in]0, \delta[\cup]\gamma, \beta[, \quad (158)$$

$$\sigma^2 x^2 w''_0(x) + bxw'_0(x) - rw_0(x) \leq 0 \quad \text{for all } x \in]0, \zeta[\cup]\alpha, \infty[, \quad (159)$$

$$w_0(x) - w_1(x) - K_0 \leq 0 \quad \text{for all } x \in]0, \gamma] \cup [\beta, \infty[, \quad (160)$$

$$w_1(x) - w_0(x) - K_1 \leq 0 \quad \text{for all } x \leq \alpha, \quad (161)$$

$$-w_1(x) - K \leq 0 \quad \text{for all } x \geq \delta \quad (162)$$

$$\text{and} \quad -w_0(x) - K \leq 0 \quad \text{for all } x \geq \zeta. \quad (163)$$

Inequality (158) for $x < \delta$ follows immediately from (149), (153) and the fact that $\delta = \ell(\gamma)$. Inequality (158) for $x \in]\gamma, \beta[$ and (159) for $x > \alpha$ hold true thanks to (147), while (159) for $x < \zeta$ is equivalent to $K \leq 0$, which is true by assumption. The inequalities (160) for $x \leq \zeta$ and (161) for $x \leq \delta$ are true because w_0 is increasing and $K_1, K_0 > 0$. Also, (160) for $x \geq \alpha$ and (161) for $x \in [\gamma, \beta]$ are both equivalent to $K_1 + K_0 \geq 0$, while (160) for $x \in]\zeta, \delta[$ will

follow as soon as we establish it for $x \in [\delta, \gamma]$ below. Furthermore, (162) and (163) follow immediately from the fact that w_1 and w_0 are increasing.

To establish (160) and (161) for $x \in [\beta, \alpha]$, we need to show that

$$-K_1 - K_0 \leq g_1(x) \leq 0 \quad \text{for all } x \in [\beta, \alpha], \quad (164)$$

where $g_1(x) = w_0(x) - w_1(x) - K_0$. Using (37), (62) and (66)–(67), we can verify that g_1 and g'_1 admit the expressions given by (86) and (87). These expressions, the fact that (161) holds with equality for $x = \beta$, and the C^1 continuity of w_1 , w_0 at β imply that

$$g_1(\beta) = g'_1(\beta) = 0, \quad g_1(\alpha) = -K_1 - K_0 \quad \text{and} \quad g'_1(\alpha) = 0$$

In view of (147) and Lemma 11 for $\nu = \alpha$, $L = -rK_1$ and $q = g'_1$, we can see that $g'_1(x) < 0$ for all $x \in]\beta, \alpha[$. It follows that $g_1(x)$ decreases from 0 to $-K_1 - K_0 < 0$, and (164) holds true.

Finally, the inequalities (160) and (161) for $x \in [\delta, \gamma]$ are equivalent to

$$-K_1 - K_0 \leq g_2(x) \leq 0 \quad \text{for all } x \in [\delta, \gamma], \quad (165)$$

where $g_2(x) = w_0(x) - w_1(x) - K_0$. Using (61)–(66) and (147), we can verify that g_2 , g'_2 admit the expressions given by (114), (115), and $g'_2(x) > 0$ for all $x < \gamma$. Combining the fact that g_2 is strictly increasing in $]\delta, \gamma[$ with the identity $g_2(\gamma) = 0$ and the inequality $g_2(\delta) \geq -K_1 - K_0$, which follows from (161) for $x \leq \delta$, we obtain (165). \square

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